LESSONS

IN

ELEMENTARY PRACTICAL PHYSICS

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PREFACE.

THE work of which this is the first volume took its origin in the felt necessity for systematising the work in our physical laboratory. We first of all commenced by using short manuscript accounts of the various processes and instruments for the use of our students. This, although satisfactory as far as it went, was found insufficient, and had to be supplemented by references to books not always accessible. Learning from various quarters the desirability of a simple yet systematic treatise on physical instruments, we were at length induced to undertake the task ourselves. It was first proposed to publish the whole in one volume, but as the matter grew under our hands it was suggested to us by Professor Sir Henry Roscoe that it might be advantageously published in three volumes. This arrangement will, we imagine, be convenient for students who wish to cultivate some particular branch of physical inquiry. The present volume will, we hope, be followed in due course by one on "Electricity and Magnetism;" and our work will be rendered complete by a third on "Heat, Light, and Sound."

It has sometimes been a cause of remark that, while natural philosophy forms a very important branch of education, there should be so few physical laboratories compared with the number of chemical laboratories that exist. One obvious reason of this is that for teaching purposes chemical processes lend themselves to a system more readily than those of physics. In a chemical laboratory, for instance, each student may have his own set of apparatus and his own place. This, of course, could not take place in a physical laboratory. Further, it has generally been thought that the expense of a physical laboratory is so great as to be almost prohibitive, except for large institutions. In this respect, however, we are of opinion that the physical will compare very favourably with the chemical laboratory. The first expenses of fitting up a complete chemical laboratory are very great; whereas, for a physical laboratory, they are comparatively small—a few deal benches and firmly-constructed slabs being all that is absolutely necessary. If, on the other hand, it be urged that physical instruments are very expensive, our reply is that this is a matter of first cost, an instrument carefully handled lasting many years, whereas in a chemical laboratory the working expenses due to breakages and the use of reagents are considerable. Nor is it necessary to commence a physical laboratory with a large number of expensive instruments, such as the dividing

engine, the cathetometer, the magnetometer, etc., for excellent substitutes may be made for some of the most expensive instruments, and constructed at little cost by the aid of a local worker in metal and wood.

It is hoped that this book may prove of use to three separate classes of students, embracing, *in the first place,* those who are attending an elementary course in a well-furnished laboratory; *secondly,* those who have access to a laboratory containing only a few instruments; and *thirdly,* those who are desirous of acquiring a knowledge of the processes of physics while they have not the opportunity of working in any laboratory. This last class will value the engravings we have given.

Our general plan has been to subdivide the work into a series of lessons, each one of these being, as a rule, descriptive of something to be done by a definite method with definite apparatus. Mathematical demonstrations have not been invariably given, our rule being to supply them only in cases where it is of importance that they should be known to the student while they are not generally found in ordinary text-books—for example, we have treated elasticity pretty fully, and the laws of motion not at all. We have attempted to make our description as clear as possible, making only a limited use of technical words.

A large portion of our present instalment consists of lessons connected with fundamental measurements,

such as those of length, mass, and density. In the latter cases, when the balance is employed, we make use of the force of gravity in order to obtain results, which are, however, independent of the local value of that force. Here we have thought it unnecessary to introduce the symbol g into both numerator and denominator of expressions from which it ultimately disappears; but we have, at the same time, in the Appendix, warned our readers against confusing *Mass* with *Weight.*

In the working out of examples and methods we have to acknowledge the services of various students in our laboratory. We especially thank Messrs. Bailey, Gerland, Harden, Jones, Kavanagh, Lees, Moss, and Turpin.

With regard to engravings, many of these were photographed from the instruments themselves, and for some help in this process in the early part of the work we beg to thank Mr. Hume-Rothery. We beg likewise to thank Mr. J. D. Cooper for the care which he has bestowed upon the preparation of the blocks, and upon his engravings from them.

Finally, for the correction of proofs we have to thank our colleagues, Professor T. H. Core and Mr. F. T. Swanwick; and for the careful verification of numerical work, a student in our laboratory, Mr. J. H. Woodward.

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PRACTICAL PHYSICS.

CHAPTER I.

Measurement of Length.

1. LENGTH, or extension in one direction, is one of the fundamental things to be measured in the Physical Laboratory.

In Great Britain the yard is the national standard of length. It is entirely an arbitrary standard—that is to say, it does not in conception or execution bear any recognised relation to any natural constant.

The **yard**[1] is defined by Act of Parliament to be the straight line or distance between the centres of two gold plugs in a bronze bar deposited in the office of the Exchequer, the temperature of the bar being 62° Fahr.

When it is convenient to use smaller units of length, the **foot**, or $\frac{1}{3}$ of a yard, and the **inch**, or $\frac{1}{12}$ of a foot, are employed. The inch is best subdivided decimally; but it is often divided into 8, 16, 32, or 64 parts.

The **mètre**, or French standard of length, was originally intended to be the 10,000,000th part of a quadrantal arc of a meridian on the earth's surface. Practically, however,

[1] See "Account of the Construction of the New Standard of Length," by G. B. Airy, Astronomer-Royal, *Phil. Trans.*, 1857; see also a treatise "On the Science of Weighing and Measuring, and Standards of Measure and Weight," by H. W. Chisholm, Warden of the Standards, *Nature Series*, Macmillan and Co.

it means the length of a certain rod of platinum at 0° C. The mètre is thus in reality an arbitrary standard, and if it should be destroyed it would be replaced by one of its copies. The mètre is subdivided decimally into 10 décimètres, or 100 centimètres, or 1000 millimètres. Its higher multiples, the décamètre, hectomètre, and kilomètre, which have the values respectively of 10, 100, and 1000 mètres, are not often required in physical work.

It is considered desirable that all scientific measurements should in all countries be expressed in terms of this metrical system.

The British Association have recommended the centimètre as the unit of length in conjunction with the gramme as the unit of mass, and the second as the unit of time. This is called the C. G. S. (centimètre, gramme, second) system (see Appendix). This system will be employed, as a rule, throughout these lessons whenever measurements involve the unit of time or the unit of mass.

2. *Relation of Metrical to English System of Measures of Length.*—The following exact comparison is taken from Dr. Warren de la Rue:—

TABLE A.

	In English inches.	In English feet.	In English yards.
Millimètre .	·03937	·0032809	·0010936
Centimètre .	·39371	·0328090	·0109363
Décimètre . .	3·93708	·3280899	·1093633
Mètre . .	39·37079	3·2808992	1·0936331

1 inch = 2·539954 centimètres.
1 foot = 3·0479449 décimètres.
1 yard = ·91438348 mètre.

To reduce millimètres to inches—

$$\text{Log mm.} + \bar{2}\cdot5951663 = \text{log inches.}$$

To reduce inches to millimètres—

$$\text{Log inches} + 1\cdot4048337 = \text{log mm.}$$

APPROXIMATE VALUES.

Mètre . .	39·37 inches; a yard and a tenth.
Millimètre .	$\frac{1}{25}$ inch; 102 mm. = 4 inches.
Yard . .	·914 mètre.
Inch . .	25·4 millimètres; 4 inches = 10·2 centimètres.

3. Whether the standard yard of England and the standard mètre of France will always accurately preserve their present lengths, and whether it might not be advisable to compare them occasionally with some natural constant, are questions which we shall not here discuss, as they are beyond the scope of these lessons. We shall take it for granted that such standards strictly preserve their constancy, so that the practical requirements of the laboratory may be regarded as satisfied when certified copies of these standards have been procured.

When such a copy is taken there is generally engraved upon it the temperature at which it is correct. The standard temperature for copies of the standard yard is about 62° Fahr., and for copies of the standard mètre about 0° C.

In accurate comparisons this point (of temperature) must not be lost sight of.

For instance, suppose that the same brass rod is divided into inches and millimètres, and that it is known to denote true inches at 62° Fahr. (16°·67 C.) and true millimètres at 0° C., it is required to find a multiplier to convert its inches into its millimètres.

We know (§ 2) that 1 true inch = 25·39954 true millimètres; but 1 inch at 62° Fahr. on brass is equal to $\frac{1}{1+16\cdot67\alpha}$ inch at 0° C., where α, or the coefficient of expansion of brass, = ·0000187. Hence at 0° C. one of its inches $= \frac{25\cdot39954}{1+16\cdot67\alpha} = 25\cdot3916$ millimètres, and as both scales expand in equal proportion, this will be the relation between the two scales at any common temperature.

Where it is wished to make use of tables which have been compiled for the conversion of inches into millimètres,

it is necessary to apply the correction now indicated if the scales are compared at a common temperature.

If the scales are of brass and correct at their respective standard temperatures, this correction is indicated by the example we have worked out. Thus we find (§ 2) that in order to convert true inches into true millimètres we have to multiply the reading in inches by 25·39954, whereas, if the comparison be made at a common temperature, the fraction is only 25·3916. When, therefore, inches are to be converted into millimètres at a common temperature on a brass scale, we shall have to multiply the tabular number by $\frac{25\cdot3916}{25\cdot39954} = \cdot99969$, or, in other words, *subtract* from the tabular number its product when multiplied by ·00031.

When millimètres are converted into inches a similar rule will apply, excepting that the correction must now be *added*.

4. A measurement of the distance between two lines drawn on a bar is known as a "line measure," or *mesure à traits*, whilst that of the distance between the ends of a bar is an "end measure," or *mesure à bouts*. The former is usually obtained by optical, the latter by mechanical means.

LESSON I.—Use of Scales.

5. *Exercise.*—Two small crosses are ruled upon a sheet of brass. It is required to measure the distance between the points of intersection.

Apparatus.—A pair of compasses (spring bows), also several scales, one divided into half millimètres, one into 64ths of an inch, a diagonal scale, and a glass millimètre scale.

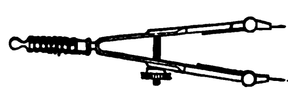


Fig. 1.—SPRING BOWS.

Method.—Apply the spring bows (Fig. 1) to the sheet of brass, so that one of its points may be in the centre of one

of the crosses, and the other of its points in the centre of the other, then apply it to the several scales. Convert all measurements into inches.

The construction and use of the diagonal scale may be understood from Fig. 2. There are eleven equidistant

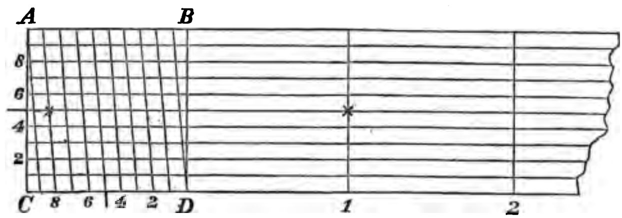


Fig. 2.—The Diagonal Scale.

horizontal parallel lines running through the whole scale, and dividing it into ten spaces. These are cut at right angles, at distances of half an inch, by vertical lines marked 1, 2, 3, etc., and by this means the whole scale is split up into a number of spaces or regions.

In the space or region at one end of the scale the lines AB and CD are divided into ten equal parts, and from the points of division diagonal lines are drawn, as shown in the figure. There will thus be two terminal triangular spaces, the sides of which are AC and BD, and nine intermediate slanting spaces. To measure a distance by means of the diagonal scale, place one point of the compass at one of the divisions, 1, 2, 3, etc., and suppose that the other point falls between two of the slanting diagonal lines, both points being in the bottom horizontal line.

Suppose, for instance, that one point is at 1, and that the other falls between 8 and 9 on the diagonal scale, then the length lies between 1·8 and 1·9. To find the length to a second place of decimals slide the compass horizontally up, keeping its right-hand point in the vertical line 1 until the left-hand point meets the intersection of a diagonal with a

horizontal line. Suppose, for instance, that when one point is at the star on the line 1, the other is at the star on the diagonal line 8 and horizontal line 5, then the measurement will be 1·85 or $=0·925$ inches, the scale being one of half inches.

The diagonal scale may be used instead of a finely-divided scale. It is ostensibly made to measure to ·0025 inch; but, as ordinarily constructed of boxwood, it cannot be depended on to this extent.

In conveying the measurements to the scales an error may be made. This may be avoided by using the glass scale and applying it directly, etched surface downwards, to the brass plate. Although only divided into millimètres it will be found easy, with the naked eye, to *estimate* to the tenth of a millimètre by this means by an imaginary subdivision into ten parts of the millimètre. This correct estimation, which is one of the first things to learn in Physical Measurements, can only be attained by practice. It will be found that, with the unpractised observer, there is a tendency to estimate the ·3 too great and the ·7 too small.

Example.—A length on a scale, divided into 64ths of an inch, was $\frac{27}{64} = ·422$ inch; on a scale divided into half millimètres it was 10·75 mm. $= \frac{10·75}{25·4} = ·423$ inch; while on a diagonal scale it was ·85 of half an inch $= ·425$ inch.

6. With ordinary scales under favourable conditions we have seen that it is possible to estimate to $\frac{1}{10}$ millimètre or ·004 inch by the naked eye. Greater accuracy may be obtained by using a sliding scale which was invented in 1631 by Pierre Vernier.[1] This device is known by the name of its inventor. The Vernier has in practice entirely superseded the diagonal scale.

[1] Pierre Vernier, *La Construction, l'usage et les propriétés du quadrant nouveau de Mathématiques.* Bruxelles, 1631.

Lesson II.—The Straight Vernier.

7. *Exercise.*—To find the length of a rod by means of a scale provided with a Vernier.

Apparatus.—A paper scale, divided into half inches, is mounted on wood, and provided with two Verniers, No. 1 and No. 2. No. 1 is 9 half inches in length, and is divided into 10 equal parts; No. 2 is 11 half inches in length, and is divided into 10 equal parts.

Method.—Place the rod AB (Fig. 3) with one end at the zero of the scale, and bring the zero of the Vernier No. 1

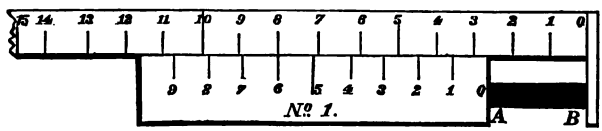


Fig. 3.—The Straight Vernier.

to coincide with the other end of the rod, as in the figure. It will be seen that the rod is between 2 and 3 units long. It will likewise be seen that 6 on the Vernier is in coincidence with one of the scale divisions; and the following simple proof will show that the true length of the rod is 2·6 units. Since 10 divisions on the Vernier = 9 divisions of the scale, therefore 1 division of the Vernier = $\frac{9}{10}$ of a scale division, or each scale division is $\frac{1}{10}$ larger than each Vernier division.

Therefore, since the coincidence is at 6 of the Vernier, the interval between

7	on the scale	and	5	on the	Vernier	= ·1	unit.
6	,,	,,	4	,,	,,	= ·2	,,
5	,,	,,	3	,,	,,	= ·3	,,
4	,,	,,	2	,,	,,	= ·4	,,
3	,,	,,	1	,,	,,	= ·5	,,
2	,,	,,	0	,,	,,	= ·6	,,

We thus know that the rod is ·6 greater than 2.

Let us now proceed to make the same measurement

with Vernier No. 2 (Fig. 4). Bring the *tenth* division of the Vernier to the end of the rod. It will again be seen that 6 on the Vernier is in coincidence with one of the scale divisions, and that the true length of the rod is 2·6 as

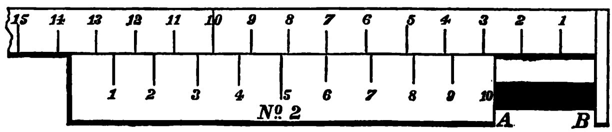


Fig. 4.—THE STRAIGHT VERNIER.

before; for in this case we have 10 divisions of the Vernier = 11 divisions of the scale, and therefore 1 division of the Vernier = $1\frac{1}{10}$ divisions of the scale, so that each scale division is $\frac{1}{10}$ smaller than each Vernier division.

Therefore, since the coincidence is at 6 of the Vernier, the interval between

6 on the scale	and	7	on the	Vernier	= ·1	unit.
5	,, ,,	8	,,	,,	= ·2	,,
4	,, ,,	9	,,	,,	= ·3	,,
3	,, ,,	10	,,	,,	= ·4	,,

We thus know that the rod is ·4 less than 3—that is to say, its true length is 2·6 as before.

8. *General Theory of the Vernier.*—It will be seen that by making n divisions of the Vernier equal to $n+1$ or $n-1$ divisions of the scale, measurements may be obtained to the $\frac{1}{n}$ of a scale division; for let L denote the length of a scale division, and V the length of a Vernier division, then first,

$$(n+1)\mathrm{L} = n\mathrm{V},$$

therefore $$\mathrm{V} = \frac{(n+1)\mathrm{L}}{n},$$

whence $$\mathrm{V} - \mathrm{L} = \frac{n+1}{n}\mathrm{L} - \mathrm{L} = \frac{1}{n}\mathrm{L};$$

or again, $$(n-1)L = nV,$$

therefore $$V = \frac{(n-1)L}{n},$$

whence $$L - V = L - \frac{n-1}{n}L = \frac{1}{n}L.$$

In the former case the Vernier reads backwards; but the difference between the divisions is more evident. The quantity $\frac{1}{n}L$ is called the "least count" of the Vernier.

LESSON III.—The Barometer Vernier.

9. *Exercise.* — To practise reading the barometer Vernier.

Apparatus. — A barometer with English scale reading to ·002 inch, or a metrical scale reading to ·05 mm. A magnifying glass.

Method. — In the *English* scale (Fig. 5) 25 divisions of the Vernier will be found to be equal to 24 divisions of the scale, whilst each division of the latter is equal to $\frac{1}{20}$ of an inch. It follows from this that one division of the Vernier is equal to $\frac{24}{25}$ of one division of the scale $= \frac{24}{25} \times \frac{1}{20} = {\cdot}048$ inch. Hence each scale division exceeds each Vernier division by $\cdot050 - \cdot048 = \cdot002$ inch. In reading the instru-

Fig. 5.—THE BAROMETER VERNIER.

ment, first read the graduation on the scale immediately below the zero point of the Vernier. This zero is a small division on the side-piece on a line with the bottom of the Vernier. The position noted in this first reading will be in inches and twentieths of an inch. Next note the place of coincidence of the Vernier with the scale. If this be at the

			Inch.	
1st	division,	then add	·002	to the previous first reading.
2d	,,	,,	·004	
3d	,,	,,	·006	
5th	,,	,,	·010	
8th	,,	,,	·016	
10th	,,	,,	·020	

And so on.

The 5th, 10th, 15th, 20th, 25th divisions of the Vernier are marked 1, 2, 3, 4, 5, which numbers represent their values in hundredths of an inch. The rule is therefore as follows:—For each large division of the Vernier ·01 is to be added to the first reading, while for each small division ·002 is again to be added.

In the *French* scale the large divisions are centimètres and the small divisions millimètres. Here 20 divisions of the Vernier will be found equal to 19 small divisions of the scale; and hence one division of the Vernier = $\frac{19}{20}$ of one small scale division = ·95 mm. Hence, also, each scale division exceeds each Vernier division by 1·0 – ·95 = ·05 mm.

In this case, too, the first reading—that is to say, the reading of the graduation on the scale immediately below the zero of the Vernier—will be in millimètres; in the next place, note as before the point of coincidence of the Vernier with the scale. If this be at the

			Mm.	
1st	division,	then add	·05	to the previous first reading.
2d	,,	,,	·10	
3d	,,	,,	·15	

And so on.

The 5th, 10th, 15th, 20th divisions are marked 25, 50, 75, 100, being their values expressed in hundredths of a millimètre.

A place of perfect coincidence may not always be found. In this case an estimation by the eye must be made as to where there would be such coincidence if the scales were further subdivided. Generally, however, it is sufficient to take the mean of the two readings between which the coincidence lies. Sometimes several divisions may appear to be in coincidence, and here the middle one should be taken.

Example, English Scale.—Looking at Fig. 5, the zero point of the Vernier on the English scale is past 30 inches and a twentieth, or 30·05. Examining the Vernier, there appear to be three points of coincidence at the 2d, 3d, and 4th small division past the large division 3. Let the 3d small division be taken as the true point of coincidence; the reading on the Vernier is therefore $\cdot 03 + \cdot 002 \times 3 = \cdot 036$ inch. We have thus—

Reading on scale	30·05
,, on Vernier	·036
Reading of barometer . . .	30·086

Example, French Scale.—The French scale (Fig. 5) reads 76·3 centimètres or 763 millimètres, and the coincidence on the Vernier may be taken as at three divisions above the large division 75; or, since each small division = ·05 mm., at $\cdot 75 + 3 \times \cdot 05 = \cdot 90$ mm. The reading is thus 763·9 mm.

10. *Comparison of Barometer Scales.—Exercise.*—If the barometer have two scales, these should be compared together. To do this, set the English Vernier exactly, using a magnifying glass, at 28 inches, 28·5 inches, etc., and then read off the corresponding values on the French scale, estimating to one thousandth of a millimètre. By multi-

plying the English readings by 25·3916 (§ 3) we should obtain the corresponding values on the metrical scale.

The following is the result of such a comparison, in which, the English scale being found correct, the object required is to obtain the error of the French scale:—

Inches.	Millimètres calculated. (1)	Millimètres observed. (2)	Difference. (2)−(1)
28	710·965	711·006	+ ·041
28·5	723·661	723·700	+ ·039
29	736·356	736·400	+ ·044
29·5	749·052	749·110	+ ·058
30	761·748	761·817	+ ·069
30·5	774·444	774·500	+ ·056
31	787·140	787·180	+ ·040
		Mean .	+ ·049

Thus the French scale is about ·049 mm. too high.

11. The accuracy of measurements made by means of the Vernier is limited by the difficulty of observing the points of coincidence when the divisions are close together. When, therefore, measures having a high degree of accuracy are required it is necessary to employ some other method.

The screw furnishes us with a ready means of accurately measuring length. We shall now, therefore, proceed to describe the Spherometer, the Micrometer Wire-Gauge, the Dividing Engine, the Micrometer Microscope, and the Whitworth Measuring Machine, in all of which the principle employed is that of the screw.

LESSON IV.—The Spherometer.

12. *Exercise.*—To measure the thickness of a thin plate of glass.

Apparatus.—A spherometer, a truly-ground and polished sheet of glass, or a Whitworth true plane. The spherometer (Fig. 6) consists of a metal tripod with three equidistant steel legs. A fine screw having a rounded point at

A, and carrying at B a graduated disc, screws through the collar to which the legs are attached. By means of an adjusting screw D the long screw may be fitted accurately, so that it does not move loosely in its bearing. A straight scale is fixed to one of the legs in such a way as to be quite near to the graduated disc.

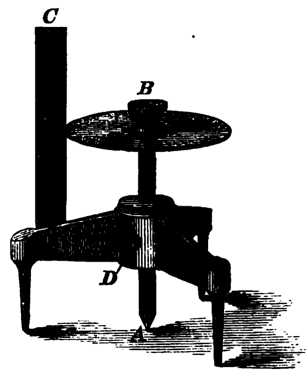


Fig. 6.—THE SPHEROMETER.

Method.—(1.) It will be found that one whole turn of the screw will raise or depress the point A through half a millimètre, and also that the circular scale B is divided into 500 parts. If therefore B be turned round through one of these parts this will denote an elevation or depression of $A = \frac{1}{2} \times \frac{1}{500} = \cdot 001$ mm. (2.) Place the instrument on the sheet of glass, or on the Whitworth plane, and turn the milled head until the central steel point A just touches the plane. When the screw has been turned a little too far the instrument may be made to hobble or rock, and in so doing to emit a peculiar noise; the screw should then be turned backwards until this noise quite ceases—a position which may be determined with great accuracy. The three legs and the point A are now all in contact with the plane. The position of this zero point should be read by finding what division of the vertical scale is opposite the disc, and also what division of the disc is next the vertical scale; and the whole operation should be repeated several times, so as to ensure accuracy in the determination of the zero. (3.) Place the piece of glass whose thickness is required under the central point A, and take readings of the position when the instrument ceases to hobble, as before.

14. *Comparison of Wire-Gauge and Spherometer.—Exercise.* —To measure by means of the spherometer and wire-gauge the thickness of a number of thin pieces of glass (microscopic object covering glasses). The following is the result of such a comparison:—

	Wire-Gauge. (1)	Spherometer. (2)	Difference. (2) − (1)
1.	0·0086 in. = 0·218 mm.	0·219 mm.	+ 0·001 mm.
2.	0·0084 ,, = 0·213 ,,	0·212 ,,	− 0·001 ,,
3.	0·0077 ,, = 0·196 ,,	0·196 ,,	0·000 ,,
4.	0·0079 ,, = 0·201 ,,	0·203 ,,	+ 0·002 ,,

LESSON VI.—The Dividing Engine.

15. *Exercise.*—To measure accurately the length of a brass bar.

Apparatus.—A dividing engine will be required. The engine that will be described is that of M. Perreaux of Paris. It is especially adapted for accurate measurement, or for dividing into any number of equal or unequal parts any given length. Fig. 8 gives a general view of the apparatus. It consists of four distinct parts—(1) A bed of cast iron; (2) An adjustable platform; (3) A micrometer screw and a travelling hollow screw; (4) The carriage, with microscope and dividing gear. We shall now describe these in order:—

(1.) *The Bed.* This consists of two parallel rods, MM and NN, accurately planed. The upper surface of MM is flat, while that of NN is angular in section, forming a long knife edge. At the end R is a metallic appendage cast in one piece with the parallel rods, and resembling in form the sides of a thick rectangular iron box, hollow within. The whole is supported by the pedestals PP fixed into a mahogany base.

(2.) *The Adjustable Platform*, SS, consists of a planed bar two inches wide. This is supported by the elon-

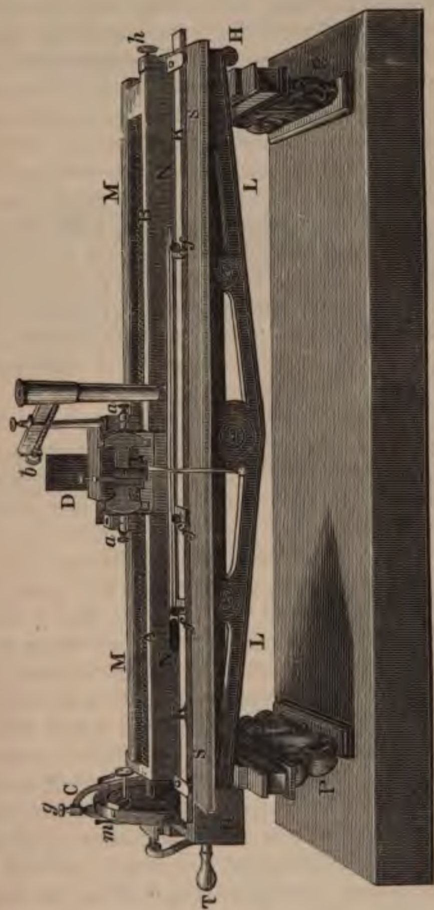


Fig. 8.—The Dividing Engine.

gated lozenge-shaped cast-iron frame LL, which is sufficiently strong to prevent any flexure of the platform. The whole arrangement is capable of travelling in a direction from or towards NN along slots attached to the upper part of the pedestals. The platform may be fixed to the required distance from NN, and adjusted to parallelism by means of the screw H. At each end of the platform is a little rectangular block of metal with a projecting flexible tongue of thin steel. A bar, KK, a little longer than the platform, is capable of having its ends placed under the tongues in a definite position, in which, when pressed home against the blocks, it becomes parallel to NN. On this bar are several little clamps *f*, for holding objects. A thermometer is shown in position for measurement.

(3.) *The Micrometer Screw*, AB, is of cast steel. It is fixed so as to be capable of revolving, having its right end hollowed out to fit a pivot, against which it may be made to bear more or less strongly by turning the milled head *h*—the other end rests in bearings. The screw passes through the centre of a brass circle C, divided into 250 parts, forming a micrometer head. This circle has a broad rim with a spiral groove running five times round it. A detailed sketch of this micrometer head is given in Fig. 9. The graduated circle has radial arms, and is supported by a central collar fitting but loosely over the screw; this circle surrounds an inner wheel E, with 250 teeth, which, unlike the outer circle, is firmly fixed to the screw. The consequence of this is that the graduated circle may turn independently of the inner toothed wheel; nevertheless a ratchet attached to the former is arranged so that it causes the circle to turn the toothed wheel, and hence also the screw when the

motion is in one direction, while, when moving in the opposite direction, the ratchet moves freely over the teeth, and the screw remains stationary. It is of importance when using the engine to be sure that

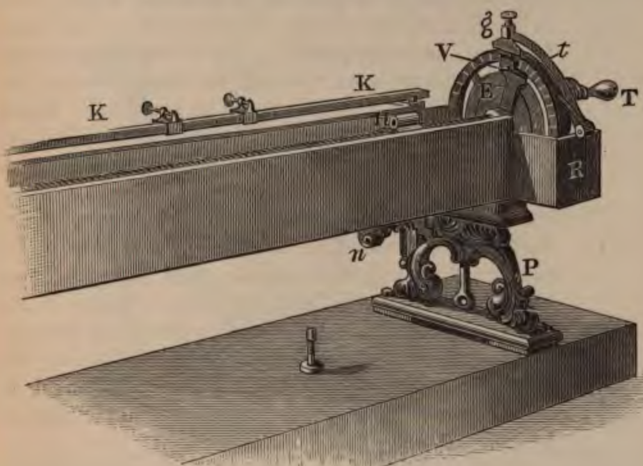


Fig. 9.—The Dividing Engine.

the ratchet works in this way—that is to say, in one direction only, and not at all in the other. In order to tell us the number of divisions through which the circle has turned, there is, bearing against its graduated face, a small brass segment V, provided with an index-mark. This index segment has another important use. The broad radial arm of which it forms a part is rather longer than the radius of the circle, and through the projecting portion a hole is drilled to hold a steel pin; this steel pin may be pushed in or out so as to cross any of the spiral lines on the rim of C, in which position

it may be clamped by a screw. The steel pin is used in connection with the *governor* g—a simple and ingenious arrangement. This is a strip of steel with a curved knife-edge, so as to lie along the spiral; it is supported by an arm t, which gives it freedom of movement. The knife-edge has three notches, one being central. The index segment may be used in two ways: it may be clamped on to the brass circle, or it may not be so clamped. When the index segment is not clamped, it is held by the central notch of the knife-edge, into which the steel pin is inserted. In this case, while the wheel turns, the segment remains as a fixed index-mark, thereby recording fractional parts of a turn. When the index segment is clamped, the steel pin may be so arranged as to be stopped by the side notch of the knife-edge when the circle has completed a definite number of turns and fractional parts of a turn. Its action in this way will be described further on in the lesson which treats of the manufacture of scales.

If the governor is not required, it may be placed to one side in the rest m (Fig. 8).

The travelling hollow screw is composed of two symmetrical hollow half cylinders united by a hinge H (Fig. 9), the two together forming a hollow screw with a thread the same as that of the micrometer screw. When the latter revolves, the travelling screw moves away from the handle T, provided that the half cylinders are pressed against the screw. This is done by arms attached to the half cylinders, by which they may be pressed together and secured by a clamp at the ends of the arms n.

(4.) *The Carriage*, D (Figs. 8 and 10), has a rectangular base with two grooves so as to fit accurately on MM and NN. A motion at right angles to the

screw is produced by the rectangular bar T, which is capable of sliding in the slot of the block W, regulated by a clamp screw X. The apparatus may be raised or lowered by unclamping Z, which allows the vertical bar T′ to slide up or down the bar W′.

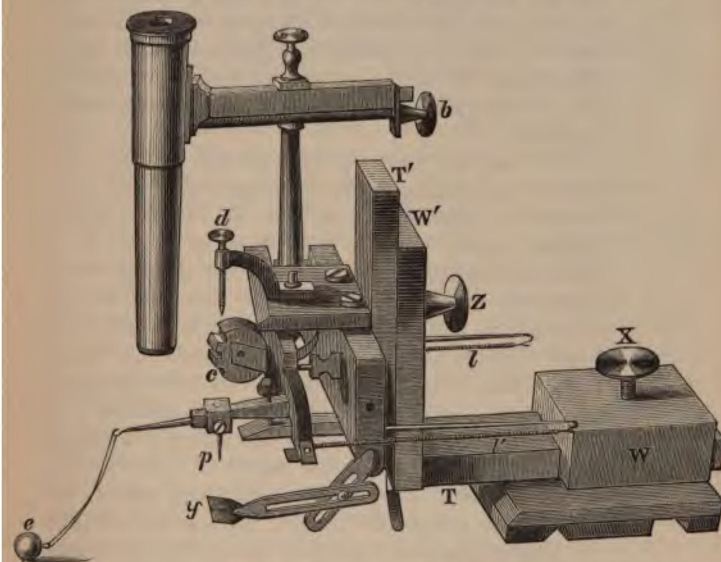


Fig. 10.—Carriage of Dividing Engine.

The travelling carriage contains three things—(α) The Microscope; (β) The Indicator; (γ) The Dividing Gear. *The microscope* is mounted so as to be capable of sliding up or down a tube supported by an arm; it is held by a vertical standard rod. The microscope has three motions: it may move in a circle by unclamping the binding screw at *a* (Fig. 8), or up and down in its supporting tube, or to and

from the standard by means of the screw-head *b* (Fig. 10), which is in connection with a long fine screw. When the carriage is in its place on the engine it fits over the projecting hinge of the travelling hollow screw, so that the two move together. To tell its position on the engine there is an *indicator* *y*, with a fine line on its index-plate. Used in connection with the indicator is a graduated half millimètre scale serving as a *counter*, and which slides in a rectangular groove on the edge of the movable platform.

The carriage contains likewise *the dividing gear*, with its scratching point, the object of which is to form with regularity the lines of division when making scales. These lines may be required of different lengths; for instance, every fifth or every tenth line may be wished longer than the others, or it may be necessary to distinguish alternate divisions. This is accomplished by a wheel with broad cogs *c* (Fig. 10), the spaces between the cogs being not of uniform depth. The wheel is mounted on a hinged plate, so that when the string *e* is drawn forward, the point of the screw *d* falls on the cogged wheel, limiting the forward play; on freeing the string the hinged plate is pulled into its former position by a pair of springs seen at *l* and *l'*. As *e* moves backwards and forwards the scratching point *p* marks the division of the required length. A wheel (on the side of *c*, not shown) with ten teeth, and a ratchet with a step by step movement, like a clock escapement, causes the cogged wheel to move through one division for each to-and-fro motion of the needle.

Method.—Fix the bar to be measured in position on the engine. Adjust the eye-piece of the microscope until the spider lines are in good focus, then slide the microscope

bodily in its holder until one end of the bar is also in focus, arranging so that the spider lines shall be parallel, the one to the graduations and the other to the edge of the bar. When the microscope is in perfect adjustment there should be no parallax—that is to say, no relative motion of the spider thread and the end of the bar caused by moving the eye backwards and forwards. The next point to guard against in this, and indeed in all machines with screws, is *loss of time* or *back lash*, which means any small turning of the screw without corresponding movement of the travelling piece. For this purpose slide the carriage so that the microscope points a little beyond the end of the bar to be measured—that is to say, nearer to the handle of the machine. Now clamp the traveller and then turn the handle until perfect coincidence between the spider line and the end of the bar is obtained. Next set the half millimètre-scale counter, so that the index-mark of y is in a line with the zero of the scale. Let the circle now be set so that its zero coincides with the index-mark on the brass segment, which is here held by the governor and not clamped to the circle. The setting to zero is performed by turning the circle backwards, an operation which does not disturb the previous adjustments. All being now in readiness for the measurement, turn the handle until the remote end of the bar comes into coincidence with the spider's thread of the microscope. The number of divisions passed over on the counter gives the whole number of revolutions, while the reading of the circle scale against the index-mark of the brass segment gives the fractional parts of a turn.

The pitch of the screw being 0·5 mm., and the circle being divided into 250 parts, each division will be equal to $\frac{1}{250} \times \frac{1}{2} = \frac{1}{500} =$ ·002 mm. or ·00008 inch. Having now made one observation, unclamp the traveller, slide the carriage back, and repeat the process.

Though the screw may have been made with great

accuracy, the pitch may not be precisely half a millimètre, and this difference may produce a sensible error when a long distance is measured. Accordingly it is necessary to determine accurately the pitch of the screw. This is done by comparison with a standard bar. Since ordinary scales are too coarse for this purpose it is necessary to have a standard bar whose lines are sufficiently finely marked to bear the magnification employed. The standard is placed on the engine, and measurements of it are made at various parts of the screw.

Example.—Measurement of a bar:—In the first place, a distance of 50 mm. on a finely-engraved standard scale was found equal to 100 turns, 52 parts, as a mean of several measurements made at various parts of the screw. Hence

$$100{\cdot}208 \text{ turns} = 50 \text{ mm.}$$
$$\text{or } 1 \text{ turn} = {\cdot}49896 \text{ mm.}$$

The error of a turn is thus ·00104 mm., which is equal to ·52 parts on the divided wheel. The standard scale was then replaced by the bar to be measured, which required 195 turns, 78 parts, and which was consequently 97·453 mm. in length.

LESSON VII.—The Dividing Engine—Manufacture of Scales.

16. *Exercise.*—To make a brass Vernier scale to enable another scale to be read to tenths of a millimètre.

Apparatus.—A dividing engine, a strip of brass with bevelled edge, wax, nitric acid, etc.

Method.—The problem is to divide either 9 or 11 millimètres into ten equal parts. Let us take the latter operation. Each division will be $1\frac{1}{10}$ mm. long. The engine will therefore require to be set so as to make 2 revolutions and 50 parts, or, allowing for the error of the engine (Lesson

VI.), 2 revolutions and 51·1 parts. The small steel pin attached to the index plate must be adjusted so that the governor travelling from the stop along the spiral groove shall allow two complete revolutions, when the index is set at zero. By moving the index to division 51·1 and clamping it there the machine is set for making divisions of the required length.

Next cover the brass with wax and fix it in position on the engine. Adjust the dividing gear so as to give marks of the required length, then, with the governor against the stop, draw forward the operating string and allow the needle in its backward journey to scratch off the wax. Now, by means of the wheel, run the carriage forward through the required distance, until the governor is caught by the pin, and again make a mark. The handle is now turned backwards until the governor is once more against the stop and the above operation repeated. If the cogged wheel has been properly set, the fifth and the tenth will be long divisions. Continue this until twenty divisions are made. Mark the central division with an arrow-head, and number the divisions, as shown in VR (Fig. 11). The scale may be fixed with nitric acid, and the wax removed. We thus obtain a double Vernier, which, when used in connection with a scale, should give readings on either side of the central line, and these readings should correspond. Thus in Fig. 11 we see that the coincidence occurs at 7 on the Vernier scale, both to the right and the left, and that the reading is therefore 11·7.

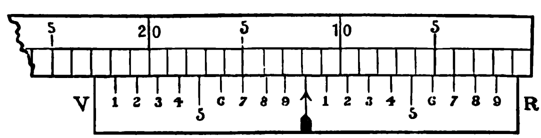


Fig. 11.—The Double Vernier.

LESSON VIII.—Copying of Scales.

17. *Exercise.*—To make a copy on glass of a millimètre scale.

Apparatus.—A millimètre scale S (a steel scale is very well adapted for the purpose); a rod of about 24 inches in length, with two needle-points; a glass tube, or slip of window-glass with ground edges; beeswax, hydrofluoric acid, spirits of turpentine, soda solution, rouge, and a gutta-percha rod.

Method.—(1) Clean the glass with soda solution; (2) warm, and evenly coat it with beeswax—this may be best done by plunging the glass into a bath of molten wax; (3) fasten the scale S and the glass T to a board or table about 24 inches apart from one another (Fig. 12); (4) insert the

Fig. 12.—COPYING OF SCALES.

point A into the first division of the steel scale, then with the point B draw a line on the wax tube so as to remove the wax. A may now be inserted into the second division, and a line again drawn with B. Continue in this manner, making every fifth and tenth division longer than the others, the tenth being also longer than the fifth. Examine the scale to see whether there are any defects; and if any of the wax has been rubbed off in places this defect must be remedied by a hot needle. Attach figures to the 10th, 20th, etc., divisions—this is best done with a fine-pointed pen which will give a double line to the figures, thus rendering them more legible. The scale may now be rendered permanent by the application of liquid hydrofluoric acid, which should be well

rubbed into the divisions by means of a gutta-percha rod —the length of time during which the acid should be allowed to remain on being determined by a preliminary experiment. When sufficiently etched the acid should be washed off, the glass dried and warmed, the wax wiped off and altogether removed by turpentine. The etched divisions may be rendered more distinct by rubbing in rouge.

It is clear that the rod really makes arcs of circles, but the radius being large the curvature is not perceptible.

This simple method of making scales is of considerable value. It has been applied by Professor Bunsen to the graduation of tubes used in gas analysis, and he has devised a convenient form of the apparatus.[1]

By removing one of the points and replacing it by a mathematical drawing-pen, scales may be made on paper suitable for hydrometers and galvanometers.

18. To measure accurately the distance between two points vertically situated requires a special contrivance— the cathetometer. This instrument was first employed by Dulong and Petit.

LESSON IX.—**The Cathetometer.**

19. *Exercise.*—To verify the length of half a mètre on a brass scale standing vertically.

Apparatus.—The instrument by which this may be done is called a cathetometer. It consists in principle of a horizontal astronomical telescope attached to a graduated vertical brass scale. This telescope, which is capable of motion up and down the scale, is furnished in its field of view with two spider threads (see Appendix), a vertical and a horizontal one. The upper object or mark is first brought upon the intersection of these cross-threads,

[1] See Bunsen's *Gas Analysis*, translated by Professor Roscoe.

which lies on the optical axis of the telescope, and in this position the reading on the fixed graduated vertical scale of a Vernier that moves with the telescope up and down is then taken. In the next place, the telescope is moved until the lower object or mark coincides with the intersection of the cross-threads and the vertical scale is once more read. The difference of these two readings will give us the difference in vertical height of the two objects or marks.

In order to obtain accuracy in this observation it is necessary that the scale to which the telescope is attached and on which it moves should always be vertical, and that the telescope should always be perpendicular to the scale. It is, moreover, obvious that as the telescope is used for viewing an object at some distance any error in the adjustment of the scale or of the telescope will increase in importance as this distance is increased. It is therefore undesirable that the object should be a very distant one. We shall suppose that the object to be measured is a graduated scale, at a distance from the telescope of about a mètre, and that we wish to measure by means of the cathetometer the length of half a mètre as seen on this scale.

A sketch of the cathetometer used for this purpose is given in Fig. 13. From this it will be seen that a cylindrical tube of brass OP, supported on a tripod stand of brass furnished with three levelling screws, serves as axis for a triangular brass prism, SS, somewhat longer than a mètre. This brass prism is furnished with a millimètre scale engraved on its flat face. A cylindrical tube W, weighted with lead, acts as a counterpoise. Thus the triangular prism, with its counterpoise, can revolve in azimuth round the tube OP, the weight of the counterpoise being such that the centre of gravity of the travelling system lies on the axis of motion, which is supposed to be vertical. The travelling system will thus move

with ease, and it may be clamped in any required position by the screw D.

A telescope, TT, is arranged to slide up and down the scale. Below the telescope is a rectangular framework of brass, one of its bevelled edges bearing a Vernier V. The telescope may be clamped in any position along the rod by F, and exactly set by the fine adjusting screw E. The telescope is supported on two forks; where the telescope rests on the forks the tube is thickened by a collar of metal, each collar being turned of exactly the same diameter. Attached to the telescope is a glass spirit-level mounted in a brass tube LL. This level is graduated, and should indicate, when the bubble is between the zero index-marks, that the axis of the telescope is horizontal. In the cathetometer (figured in Fig. 13) the

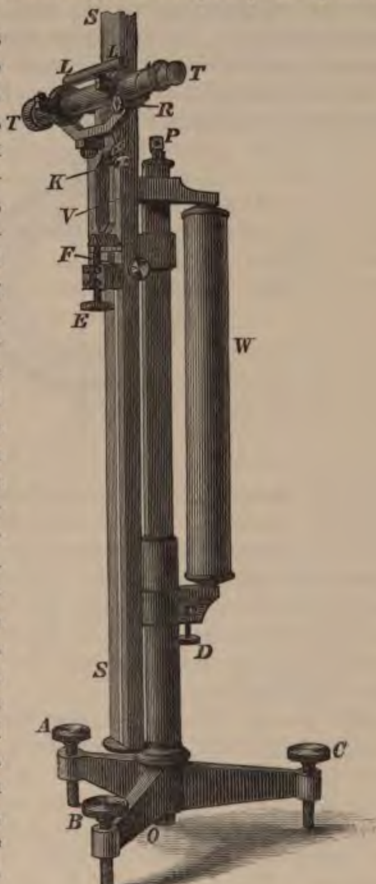


Fig. 13.—The Cathetometer.

maker has once for all secured this condition, but by

attaching the level to the telescope and providing the level with an adjusting screw we shall have this adjustment under control.[1] The telescope may be focused by the screw R, and levelled by the screw K.

In the focus of the eye-piece are the spider lines already mentioned crossing the centre of the field of view at right

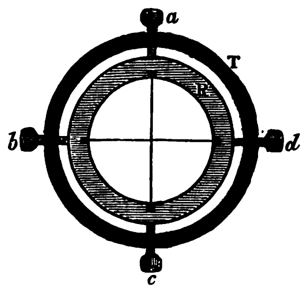


Fig. 13*a*.—RING WITH CROSS-THREADS.

angles to each other. The cross-wires are fixed on a disc of metal somewhat smaller than the telescope-tube. It is held in position by four small screws, which permit the exact adjustment of the intersection of the cross-wires to the axis of the telescope (see Fig. 13*a*).

The instrument ought to stand on a block of stone firmly fixed to the earth and free from the disturbing effect of vibrations.

Method.—It will *first* be necessary to ascertain whether the intersection of the threads lies in the optical axis of the telescope. *Next*, to see whether, when the bubble of the level lies between its marks, the axis of the telescope is horizontal. The *third point* is to secure the verticality of the scale and the horizontality of the telescope in all positions. The *fourth point* is to secure a distinct tele-

[1] Or independently of the level this adjustment may be made in the manner described in the Appendix, where additional details relating to the cathetometer adjustments will be found.

scopic image of the mark and of the cross-threads, these being such that when the eye is moved backwards and forwards, or up and down, there shall be no parallax—that is to say, no apparent motion of the mark with reference to the cross-threads. In this case the image of the mark given by the object-glass of the telescope is exactly in the plane of the cross-threads, and hence, when viewed through the eye-piece, there can be no relative motion of the two (mark and threads) any more than there can be of the letters of this page with reference to the page itself. It is only when things are at different distances that there can be a relative motion of the one with regard to the other when the eye is moved about.

I. To bring the intersection of the cross-threads into the optical axis of the telescope. Focus upon some mark, then, everything else remaining undisturbed, rotate the telescope about its own axis. If the centre of the cross-threads should not remain on the mark, the four screws, *a*, *b*, *c*, *d*, of the supporting ring (Fig. 13*a*) should be altered until this constancy is obtained.

II. To secure that the axis of the telescope shall be horizontal when the bubble of the level lies between its index-marks. Level the telescope by means of the telescope-screw. Then remove the telescope with its attached level, turn it end for end, and replace it on the supporting forks. If the bubble is not now between its index-marks, take half the error out by means of the screw which is attached to the level. Let this operation be repeated several times.

III. In order to secure the verticality of the scale and horizontality of the telescope, the following adjustments must be made :—

(1.) Turn the telescope so as to be parallel to the line BC that joins two of the levelling screws, then level by means of K.

In order to make the axis vertical and the telescope horizontal, let us suppose that we are looking down upon the instrument from above, and that A, B, C (Fig. 14a) are the foot-screws, and P′ the projection of the upper point P of the axis OP on a horizontal plane. When the instrument is in perfect adjustment P′ should coincide with O. We shall first bring P′ into the vertical plane passing through AO—that is to say, into the position P″—and then by adjusting A move P″ along AO till it comes to O.

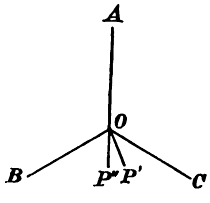


Fig. 14a.

(1.) The first point, then, is to bring the axis into the plane through AO, the telescope being perpendicular to the axis. To perform this, place the telescope parallel to BC (perpendicular to AO) and make its level horizontal. Next turn the instrument through 180°, and if the telescope is not now level, half the error is due to the telescope level not being perpendicular to the axis, and half to the axis being out of AO. This will be seen from Fig. 15, for if TEL be the telescope levelled in the first position, and we then turn it through 180°, it will take a position T′EL′ such that T′EO = TEO. But TEO = PEL. Hence T′EO = PEL. Hence if we draw NEN′ perpendicular to OP it must bisect T′EL. The axis, therefore, is out of the vertical just as much as NEN′ is out of the hori-

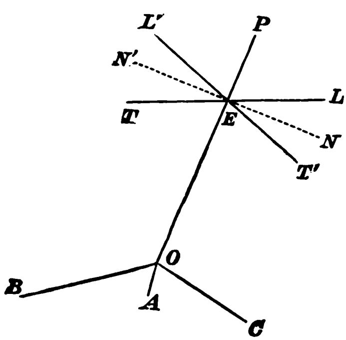


Fig. 15.

zontal—that is to say, by the angle NEL = ½ T′EL. If, therefore, we correct half the error by means of the *levelling* screw K, we shall have the telescope perpendicular to the axis; and if we correct the other half by means of the *foot* screw B or C, the axis will now be in the plane through AO.

(2.) To bring P″ vertically over O turn the telescope parallel to AO (Fig. 14*a*), and adjust the screw A until the level is horizontal. It should now be horizontal in all positions.

21. *Modification of the Cathetometer.* — In order that measurements taken by a cathetometer should be trustworthy, the workmanship should be of the very best kind, many instruments not being trustworthy to ·1 mm., although graduated to read to ·02 mm. The method of supporting the instrument on three legs is convenient for general purposes, but it is far better to support it against a wall of masonry, the telescope being movable by means of a pulley. This method has been followed at the Kew Observatory.

Lesson X.—The Micrometer Microscope.

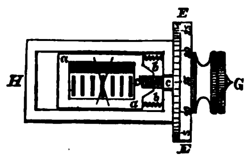
22. When lengths are to be compared which differ by a very little from one another, and generally when small lengths are to be measured, it is convenient to use the micrometer microscope. This instrument can measure quantities so minute that there may be some difficulty in eliminating sources of error such as those due to expansion by heat and change of shape by flexure. It was used by Airy in his reproduction of the standard yard, on which occasion he employed a microscope capable of reading to ·000025 inch.

It (Figs. 16 and 17[1]) consists of a microscope (Fig. 17)

[1] Taken from Chauvenet's *Astronomy*.

with a Ramsden[1] eye-piece AB, provided with a micrometer HG. A plan of the micrometer (with the image of the divisions of a scale, under measurement, in its focus) is seen in Fig. 16. A fine screw *cc* (attached to a milled head G, and a divided circle EE) moves a rectangular framework *aa*, which carries the cross-wires. At one side of the framework, but not movable, is a scale, the distance between its divisions being equal to the pitch of the screw. Two spiral springs *bb* make the screw always bear on the same side of the thread so as to avoid *loss of time*. It will be understood that measurements are made by noting the number of revolutions and parts of a revolution required to pass the centre of the cross-wires from the one end to the other of the length under measurement.

Fig. 16.



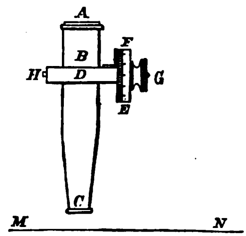


Fig. 17.
THE MICROMETER MICROSCOPE.

LESSON XI.—**The Whitworth Measuring Machine.**

23. The principle of measurement involved in the micrometer wire-gauge and spherometer, in which the sense of touch is made use of, has been developed to a high state of perfection by Sir Joseph Whitworth in his measuring machine. Imagine a wire-gauge of large size, the teeth being replaced by metal blocks whose surfaces are true planes, and accurately perpendicular to the axis of motion.

[1] Eye-pieces are of two kinds, the Positive or Ramsden, and the Negative or Huyghens. The former is the one usually employed for measuring operations. See Glazebrook's *Physical Optics*, chap. iv.

One of these blocks is capable of movement by means of a large micrometer head divided into 250 parts. Each revolution of this head causes a worm-wheel to advance $\frac{1}{200}$ of a turn, whilst a whole turn of this worm-wheel moves by means of a screw the movable block through $\frac{1}{20}$ of an inch. Each micrometer division will thus have a value of $\frac{1}{20} \times \frac{1}{200} \times \frac{1}{250} = \frac{1}{1000000}$ of an inch. Thus the difference of one millionth of an inch in the length of a bar may be recognised. The Whitworth machine is of the very highest value in obtaining "end" measurements.[1]

24. The ordinary compound microscope is convenient for determining the size of small objects.

LESSON XII.—The Microscope—Use of Glass Micrometer.

25. *Exercise.*—To find the diameter of a capillary tube.

Apparatus.—An ordinary compound microscope[2] with objectives of low power. A stage micrometer, an eye-piece micrometer, slip of glass and wax.

The stage micrometer consists of a plate of glass with fine lines ruled on it by means of a diamond, the distance between the lines being known. The eye-piece micrometer consists also of a ruled slip of glass, but in it the lines are much coarser; it is placed at the focus of the eye-piece, and as the eye-piece is usually negative, this enables it to rest on a diaphragm between the two lenses. It will then be in good focus for normal sight.

Method.—Focus for the lines on the stage micrometer, and then compare them in the field of view with those of

[1] For a full description of the Whitworth machine, see *The Whitworth Measuring Machine*, by Professors Goodeve and Shelley: Longmans & Co.

[2] For a description of the microscope and general manipulation with it, see *Carpenter on the Microscope.*

the eye-piece micrometer. The value of a division of the latter will thus become known.

Cut off a small portion of the capillary tube so as to secure a clean cross-section. If the capillary tube be thick-walled this may be done without difficulty by simply filing the tube at the place of desired section, which may then be readily broken across. But if the tube be thin-walled some care is necessary: the tube should be held between the thumb and forefinger, the latter forming a cushion for the delicate tube; a very fine file with a *knife-edge* should then be applied with little pressure.

Mount the fragment of tube vertically on the glass slip with wax. In doing so, advantage may be taken of the reflexion of the tube in the polished surface of the slip of glass, which should coincide with the tube in two directions at right angles to each other. Then place the glass slip in position on the microscope, arrange the illumination and focus until the walls of the capillary tube are well defined. If the bore be approximately circular, read the number of micrometer divisions which it covers in several directions, and the mean may be taken as the required diameter; but on the other hand, if it be distinctly elliptical, as in the majority of the thermometer-tubes, measure the major and minor axes only. From these measurements the area of the circle or ellipse can be obtained by calculation if required.

Example.—A stage micrometer with lines $\frac{1}{100}$ of an inch apart was used, and 60 spaces of the eye-piece were found to correspond with three spaces of the stage micrometer. One division of the eye-piece micrometer is therefore equal to $\frac{1}{2000}$ of an inch, or ·00127 centimètres. A tube elliptical in section gave major axis $a = 17$ divisions $=$ ·0216 cm., minor axis $b = 9{\cdot}5$ divisions $=$ ·0120 cm. Hence area $= \frac{\pi}{4} \times ab = \frac{3{\cdot}1416}{4} \times {\cdot}0216 \times {\cdot}012 = {\cdot}0002036$ square centimètres.

Lesson XIII.—General Review of Length Measurers—Special Instruments.

26. We have seen that in order to find the distance between two given points A and B we should use a divided rule as our standard. The measurement then consists simply in the comparison of the distance AB with the rule. To effect this comparison the simplest method is to apply the rule directly to the given distance. By this means, taking care to avoid parallax by the use, for instance, of the glass scale alluded to in Lesson I., measurements may be made correctly to the tenth of a millimètre. However, many cases occur where the direct application of the rule is not possible, the points A and B not being accessible to the rule. For example, it may be required to find the diameter of a sphere. In such a case we must by some artifice make other two points C and D approach each other until they touch the sphere at the ends of its diameter, and thus the length is transferred into a condition adapted for measurement. We have length-transferring instruments of this kind in the Compass, Callipers, and the Beam Compass.

27. Callipers (Fig. 18) are specially employed for measuring the external or internal diameters of curved bodies. The *Outside Callipers* constitute a compass with curved legs. The points must be set so that they just glide over the cylinder or other body to be measured, and they are then applied to the rule. The *Inside Callipers* are used in a similar manner to find the internal diameter of a hollow cylinder, hemisphere, etc. The tool is introduced into the cavity and the points set as before. Fig. 18 shows the two kinds combined in one instrument. In the compass

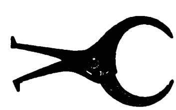


Fig. 18.—Callipers.

(Fig. 1), as well as in the callipers, the distance between the points is adjusted by aid of a *joint.*

28. In the **Beam Compass** this adjustment is made by a *slide.* This instrument (Fig. 19) consists of a straight rod of wood or metal with two adjustable points which must first be applied to the object of measurement and

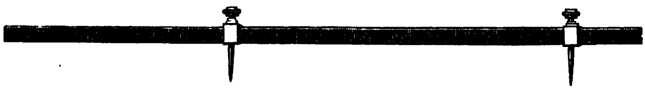


Fig. 19.—BEAM COMPASS.

then to the scale. The beam compass is very convenient when the points to be measured are some distance apart and not very accessible. The accuracy of the instrument may be increased by giving it a divided scale on its rod and a Vernier connected with the adjustable points. It is obvious that the method of sliding used in the beam compass may be adapted to the calliper.

We then have the **Slide Calliper,** a useful instrument when graduated and provided with a Vernier, far better fitted for accurate measurements than the ordinary workshop tool.

The **Micrometer Wire-Gauge** (Lesson V.) is evidently a calliper on the *screw* principle.

29. Tools of the type of the calliper are only suitable for "end" measurements, where actual mechanical contact may be obtained. In many cases this is impossible, and we must then employ the optical method, in which an imaginary line—the axis of the telescope or microscope—is successively applied to the points A and B, the amount of movement of the optical system being measured by a Vernier or screw. We have seen how this method is applied in the dividing engine, but we are unable to use this engine for the measurement of great lengths, for besides being tedious

under such circumstances it might admit of an accumulation of errors. Indeed, the screw is unsuitable for the measurement of great lengths, and an instrument on the slide principle may here be employed with advantage, a screw being, however, used to give the final adjustment.

30. A convenient instrument of this kind consists of a substantial horizontal bar about a mètre in length, having a millimètre scale. It is mounted on levelling screws, and has a microscope, carrying a Vernier, and capable of sliding along the bar. The cross-wires of the microscope are first brought to coincide with the point A and then with the point B, and the difference of readings gives the required length. The instrument thus resembles a cathetometer placed horizontally, with a microscope in place of a telescope. A simple arrangement of this nature has been employed for the calibration of thermometers by R. Brown.[1] It consists of a microscope fitted so as to slide on a board half a mètre long. The position of the microscope is read off by means of a Vernier, reading to ·1 mm.

31. By using two small compound microscopes capable of sliding on an iron bed like that of the dividing engine we have a kind of optical beam compass very suitable for many measurements. The microscopes may have cross-wires in the focus of their eye-piece—or, better still, glass scale micrometers. They are supposed to look down on a platform, upon which the object to be measured is placed, and they are adjusted until the points to be measured are brought into focus. They are then clamped in these positions, and the object is removed and replaced by a divided scale, taking care meanwhile that the microscopes remain undisturbed. The divisions of the scale which come under the centre of the microscopes may now be read, provided the scale is in good focus; if not, the scale must be raised

[1] See *Phil. Mag.*, vol. xiv. p. 57 (1882).

or lowered until accurate definition is secured, the microscopes themselves remaining undisturbed.

32. The principle of comparison by substitution just now described has been brought to a very high state of perfection in the "**Comparateur**," an instrument used for the comparison of standards.[1] Here the difference in length between the objects to be compared being but small, the microscopes may be rigidly fixed and provided with the screw micrometer described in Lesson X., an arrangement which will give the comparison a very great amount of accuracy. Elaborate mechanism is employed for the substitution of the bars under comparison, and for keeping them at a constant temperature.

33. The cathetometer, which is an instrument on the slide principle, has many defects and is difficult to work with. It is therefore desirable to abandon its use whenever possible, especially when the total vertical height to be measured is but small, for then the percentage of error through its defects will be so large as entirely to vitiate the results obtained. An excellent instrument to use in such a case is a horizontal microscope with a vertical glass micrometer. We shall describe the instrument as used by Professor Quincke of Heidelberg.[2] This **Cathetometer Microscope** (Fig. 20) consists of a wooden stand or table with levelling screws. The table A may be raised or lowered

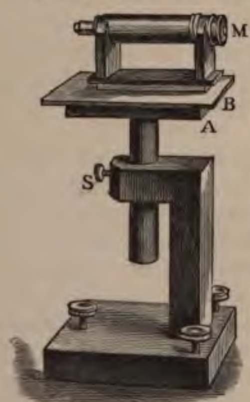


Fig. 20.
The Cathetometer Microscope.

[1] For a description of the *Comparateurs* used at the International Bureau of Weights and Measures, see *Nature*, September 13, 1883.
[2] Wiedemann's *Annalen der Physik und Chemie*, Band. xix.

by unclamping the screw S. On A is cemented a glass plate B. The microscope M is fixed in two Y-shaped wooden supports, the whole having a glass base, thus enabling the microscope to be freely moved about on the glass-covered table (glass on glass), the motion being brought better under control by dusting the table with a little lycopodium.

The microscope has an eye-piece micrometer with a scale, which may be used either in a vertical or a horizontal position by simply rotating the microscope. In using the instrument it is first levelled by the aid of a circular spirit-level placed on the table. The value of a micrometer division is next ascertained by focusing the instrument upon a glass scale placed vertically. Its application to the measurement of several constants will be subsequently described.

34. A **Reading Telescope** is often of great assistance in obtaining vertical heights. It consists simply of an ordinary telescope with a horizontal cross-wire, capable of sliding up and down a vertical rod. The length to be measured is placed by the side of a graduated millimètre scale. The cross-wire is first brought to coincide with the upper mark, the scale being at the same time visible in the field of the telescope. The reading of the scale corresponding to the cross-wire is then taken; the same is then repeated for the lower mark, and the difference of the two readings will give the required height. Since a slight difference in level of the telescope will not affect the accuracy of the result, this method should be used in preference to the cathetometer whenever possible.

35. In verifying measuring instruments assistance may be obtained from the standards introduced by Sir Joseph Whitworth, which consist of cylinders of steel of known dimensions.

The diameters of wires and the thicknesses of metal plates are in commerce specified by a number known as the wire-gauge. Until August 1883 there was no legal

wire-gauge, so that to know the number of a wire gave but uncertain information of its diameter. The new gauge, however, it is hoped, will become of general use. We give its values in English and French measure :—

TABLE B.

THE ENGLISH WIRE-GAUGE.[1]

No. on New wire-gauge	Diameter.		Area of cross-section.	No. on New wire-gauge	Diameter.		Area of cross-section.
	Inches.	Centimètre	Sq. Centimètre		Inches.	Centimètre	Sq. Centimètre.
7/0	·500	1·270	1·267	23	·024	·0610	·00292
6/0	·464	1·179	1·091	24	·022	·0559	·00245
5/0	·432	1·097	·946	25	·020	·0508	·00203
4/0	·400	1·016	·811	26	·018	·0457	·00164
3/0	·372	·945	·701	27	·0164	·0417	·00136
2/0	·348	·884	·614	28	·0148	·0376	·00111
0	·324	·823	·532	29	·0136	·0345	·000937
1	·300	·762	·456	30	·0124	·0315	·000779
2	·276	·701	·386	31	·0116	·0295	·000682
3	·252	·640	·322	32	·0108	·0274	·000591
4	·232	·589	·273	33	·0100	·0254	·000507
5	·212	·538	·228	34	·0092	·0234	·000429
6	·192	·488	·187	35	·0084	·0213	·000358
7	·176	·447	·157	36	·0076	·0193	·000293
8	·160	·406	·130	37	·0068	·0173	·000234
9	·144	·366	·105	38	·0060	·0152	·000182
10	·128	·325	·0830	39	·0052	·0132	·000137
11	·116	·295	·0682	40	·0048	·0122	·000117
12	·104	·264	·0548	41	·0044	·0112	·0000982
13	·092	·234	·0429	42	·0040	·0102	·0000811
14	·080	·203	·0324	43	·0036	·00914	·0000657
15	·072	·183	·0263	44	·0032	·00813	·0000519
16	·064	·163	·0208	45	·0028	·00711	·0000397
17	·056	·142	·0159	46	·0024	·00610	·0000292
18	·048	·122	·0117	47	·0020	·00508	·0000203
19	·040	·1016	·00811	48	·0016	·00406	·0000130
20	·036	·0914	·00657	49	·0012	·00305	·00000730
21	·032	·0813	·00519	50	·0010	·00254	·00000507
22	·028	·0711	·00397				

[1] Taken from the *Board of Trade Circular*.

It will be seen that the new gauge is considerably different from the Birmingham wire-gauge as given in Table B_1 :—

TABLE B_1.

THE BIRMINGHAM WIRE-GAUGE.[1]

B. W. G.	Ins.	B. W. G.	Ins.	B. W. G.	Ins.	B. W. G.	Ins.
No. 1 =	·312	No. 10 =	·137	No. 19 =	·042	No. 28 =	·014
2 =	·284	11 =	·125	20 =	·035	29 =	·013
3 =	·261	12 =	·109	21 =	·032	30 =	·012
4 =	·239	13 =	·095	22 =	·028	31 =	·010
5 =	·217	14 =	·083	23 =	·025	32 =	·009
6 =	·208	15 =	·072	24 =	·022	33 =	·008
7 =	·187	16 =	·065	25 =	·020	34 =	·007
8 =	·166	17 =	·056	26 =	·018	35 =	·005
9 =	·158	18 =	·049	27 =	·016	36 =	·004

To obviate the uncertainty caused by the multitude of gauges, it is convenient to express the diameter of a wire in mils., a mil. being defined to be the thousandth of an inch, thus a wire of No. 23 B. W. G. would be, according to Table B_1, a wire of 25 mils.

The approximate thickness of a wire may be readily known by using a sheet-metal gauge (Fig. 21), which consists of a metal plate with a graduated series of notches, each notch being numbered according to some specified table of wire-gauges. It is only necessary to ascertain the number of the notch into which the wire will just fit, and then a reference to the table will give the diameter.

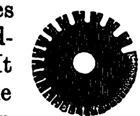


Fig. 21.
WIRE-GAUGE.

36. The student should now be in a position to devise for himself any length-measuring instrument that may be required in special cases.

Some important indirect methods of measurement of length are given in connection with the measurement of angles.

[1] Taken from Molesworth's *Engineering Formulæ.* On comparison with other Birmingham wire-gauge tables the above will be found to differ mainly in the values given from No. 1 to No. 10.

CHAPTER II.

Angular Measurements.

37. UNITS OF ANGULAR MEASUREMENT.—The ordinary unit is the **degree**, which is divided into 60 *minutes*, each minute being again divided into 60 *seconds*. The degree is the angle formed by two radii of a circle which enclose $\frac{1}{360}$ part of the whole circumference. Degrees, minutes, and seconds are written thus, $83° \ 15' \ 32''$. Another method of expressing angles is in circular measure. Here the unit is the angle subtended by an arc of a circle equal in length to its radius. Such an angle is sometimes called a **radian.** According to this method the circular measure of 180° is π, so that in order to convert an angle of $n°$ into circular measure we should multiply n by π and divide by 180.

38. *The Dividing of Circles.*—Most instruments for measuring angles are provided with a graduated circle. To graduate a circle it is generally first of all divided into six equal parts of 60° each. Each of these is then twice bisected, giving intervals of 15°, which are ultimately divided by a method of trial and error into degrees. As far as laboratory practice is concerned, we shall assume that it is always possible to make use of a circle already graduated, the question being to copy its divisions. A sketch of the method employed for this purpose is given in the following Lesson.

LESSON XIV.—Copying of Circular Divisions.

39. *Exercise.*—To divide into degrees a circle of cardboard.

Apparatus.—The apparatus to be used (Fig. 22) consists of a brass circle divided into degrees, and having a radial arm capable of turning about the centre of the circle.

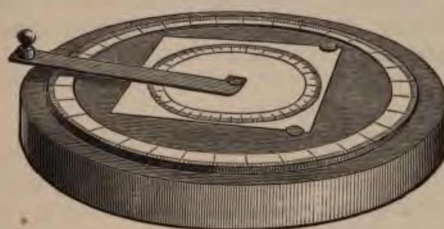


Fig. 22.—GRADUATION OF CIRCLE.

One edge of this arm is bevelled, and this edge moves exactly as a radius of the circle. Cardboard, drawing-pen, Indian ink, etc., are likewise necessary.

Method.—Remove the radial arm, and fix the stout pin about which it revolves through the centre of the cardboard circle. Replace the arm, which now has the circle beneath it, and by means of drawing-pins prevent the circle from moving. Now bring the radial arm close to the zero of the brass scale, leaving just room for the drawing-pen; then, whilst the arm is held firmly, rule a division on the cardboard. In this way the divisions of the outer brass circle are transferred, the 5th, 10th, etc., divisions being made somewhat longer than the others. The success of the operation depends upon keeping the relative position of the drawing-pen and the ruler the same at each ruling.

40. The circles of philosophical instruments are gradu-

ated by means of the "copying" method now described. Instead, however, of moving the radial arm, this may be kept fixed, both circles being mounted so as to revolve together, the motion being given by a tangent screw. This arrangement is the one commonly adopted.

41. In order to read off an angle on a divided circle with accuracy, it must be provided with a Vernier. The theory of the circular Vernier is identical with that of the straight Vernier already given. We now proceed to describe various kinds of circular Verniers.

LESSON XV.—The Circular Vernier.

42. *Exercise.*—To learn how to read various circular Verniers, such as those found on the circles of the telescope, the goniometer, and other instruments involving the measurements of angles.

Apparatus.—We shall confine ourselves to the two following Verniers: (I.) the Vernier reading to minutes, (II.) that reading to 20 seconds (20″).

Method (I.)—On examining the divided circle we shall suppose that each degree is found to be divided into 3 equal parts or spaces of 20′ each, and that 20 divisions on the Vernier are found to be equal to 19 scale divisions. One Vernier division is therefore equal to $\frac{19}{20} \times 20'$ or 19′, and the difference between one division of the scale and one of the Vernier is 1′, which is the *least count.* Fig. 23 shows the Vernier and a portion of the scale. From this we learn three things—(1) that the zero point of the Vernier, indicated by the arrow, lies between 8° and 9°; (2) that it falls beyond the second of the three parts into which each scale degree is broken up; (3) that the coincidence between the scale and the Vernier divisions is at 4. The reading is therefore 8° 44′.

(II.) Here (Fig. 24) each degree is divided into four

parts of 15′ each, and 45 divisions on the Verniers are equal to 44 on the scale. One division of the Vernier is

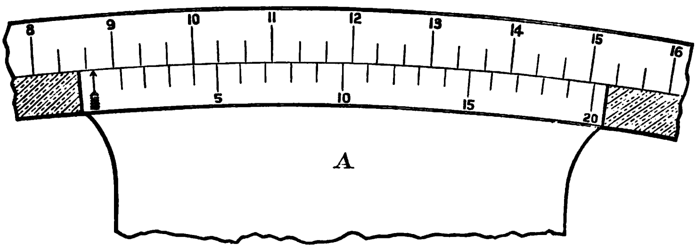


Fig. 23.—CIRCULAR VERNIER.

thus equal to $\frac{44}{45} \times 15$ or $14\frac{2}{3}'$, and the *least count* is $15' - 14\frac{2}{3}' = \frac{1}{3}'$ or 20″. It will take, therefore, three such differences to make one minute. In order to read the angle, observe the position of the zero mark of the Vernier. In Fig. 24, which shows a portion of such a Vernier

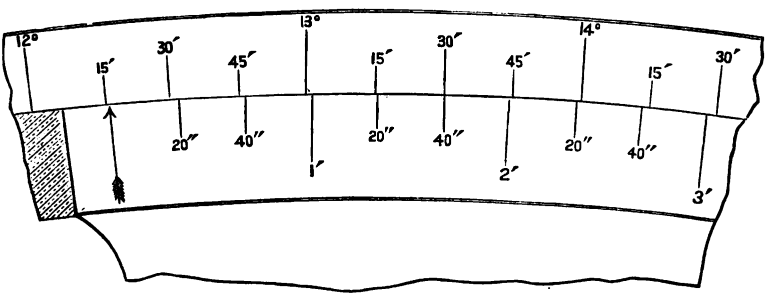


Fig. 24.—CIRCULAR VERNIER.

drawn on a large scale, the zero of the Vernier indicated by the arrow is beyond 12° 15′, and the coincidence between the two scales appears to be at the second small division beyond the 1′ division, or at 1′ 40″. The reading is therefore 12° 16′ 40″.

In order to assist in reading the Vernier a small microscope is generally provided, known as a reading microscope. The Vernier is often placed at an angle with the scale, and not in the same plane with it. This increases the difficulty of reading, introducing a possible error of parallax. Moreover, circular Verniers are very seldom provided with any means of illuminating their fine scales, so that great fatigue of the eye is often experienced in reading them. The makers of instruments, with few exceptions, do not appear to have given any attention to this point.

43. *Use of two Verniers.*—All exact instruments are provided with more than one Vernier. Usually there are two placed at an angle of 180° with each other, distinguished as "Vernier A" and "Vernier B." The purpose of the second Vernier is to neutralise the error which would arise if the centre about which the Vernier rotates should not be exactly coincident with that of the graduated circle. Thus, in Fig. 25, if C′ be the centre of the divided circle, and C that of the Vernier when a reading is taken, the one Vernier is at A instead of A′, and the other at B instead of B′. If we call x the true reading and E the error of the Vernier, while A and B are the Vernier readings, then

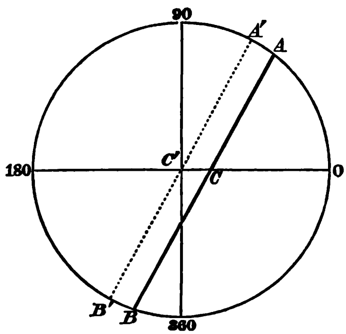


Fig. 25.

$$x = \mathrm{A} + \mathrm{E} \text{ for the upper Vernier,}$$

also for the lower $x + 180 = \mathrm{B} - \mathrm{E}$.

Hence $$2x + 180 = \mathrm{A} + \mathrm{B} \text{ and } x = \frac{\mathrm{A} + (\mathrm{B} - 180)}{2}.$$

Thus, after deducting 180° from the reading of Vernier B,

if we take the mean between this and the reading at A the true reading will be obtained, and the error due to eccentricity eliminated.

44. *The Reading Microscope with Micrometer.*—A far more refined method is to use, instead of a Vernier, a fixed micrometer microscope similar to that already described in Lesson X. Let us suppose that the circle, with its attached telescope, is made to revolve, and that the micrometer microscope, remaining fixed, is focused upon the divided scale of the circle, the cross-thread being at the centre of the field denoted by the central division of the fixed micrometer scale (see Fig. 16). If the cross-thread falls on an exact division of the circle, all that is necessary is to take its value, but, if it does not so fall, we must turn the micrometer head until the cross-thread is carried to the nearest small scale division. For instance, let us suppose that the direct reading is 29° 35′, and that the micrometer head has 60 divisions on it, a whole turn representing 1′, so that one division will represent 1″; also let us suppose that we have to turn the head through 3 revolutions and 25·3 divisions before the cross-wire can be brought to the above reading, then this micrometer motion will represent 3′ 25·3″, and the true reading will be 29° 38′ 25·3″, on the supposition that the micrometer reading has to be added to the other.

45. *The Measurement of Small Angles.*—There are various methods of measuring small angles, such as—(1) that by the filar micrometer; (2) that by the optical lever; (3) that by the mirror and scale; (4) that by the level.

46. *The Filar Micrometer.*—This instrument is much used by astronomers for obtaining the angular distance between two stars that are very near together. The filar micrometer resembles the reading microscope with micrometer, but, in addition to the movable thread, there is

also a fixed one. The movable thread is fitted on a slide in connection with the micrometer, so that it may pass in front of the fixed thread. To obtain an angular measurement by this instrument, one object or star is brought upon the fixed central cross-thread, while the movable thread is likewise brought into coincidence with the same object. The micrometer having been read, the head is turned until the thread comes into coincidence with the second object or star, and the reading once more taken. We thus obtain a measure of the desired angle, in terms of the divisions of the micrometer. To convert this into angular measure the following method may be employed:—

Ascertain (see Vol. III.) the focal length of the object-glass; let this be called F, and let p denote the pitch of the micrometer screw, namely, the linear value of one turn, then if θ (the angular value of one turn) be very small,

$$\tan\theta = \frac{p}{F}.$$

47. A very refined method for angular measurements is that based upon the laws of reflexion in optics, which is also frequently of great service in the indirect determinations of small lengths, such as the measurement of the increase in length of a rod when heated. We proceed now to explain the use of the *optical lever*, an instrument designed by M. Cornu, in which this principle is employed.

Lesson XVI.—The Optical Lever.

48. *Exercise.*—To find the thickness of microscope covering glasses.

Apparatus.—An optical lever, Fig. 26. This is a rod, provided at its centre with a mirror mounted so as to be capable of rotation about an axis, HR. The lever is supported on four steel legs, and of these a is adjustable

with respect to height. The instrument is supposed to stand on a Whitworth true plane, or a piece of perfectly

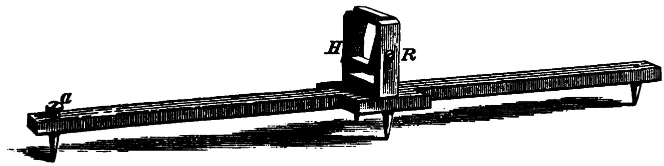


Fig. 26.—The Optical Lever.

true mirror glass, and the leg *a* is adjusted until not the slightest tendency to hobble is perceived.

Besides this, we must have a small telescope mounted on a stand (such as a cathetometer or a reading telescope), provided with cross-wires, and having a paper millimètre scale fixed vertically above the telescope in the region of the cross-wires.

Method.—Place the lever on its plane, so that a line passing through the front and back legs may be in the prolongation of the axis of the telescope. Focus the telescope (which should be several mètres distant) upon the mirror, and gradually alter the focus until some object near the telescope is seen in the telescope reflected from the mirror; then move the lever, and tilt the mirror until the divisions of the scale are so seen. These adjustments having been made, place under the central legs of the lever the object to be measured, and read off on the telescope the division of the scale which coincides with the cross-wire, one end of the lever being now raised above the plane, and the instrument standing on three legs. Next, tilt the other end up, placing a small weight on the end previously raised, and again read the scale.

Let d (Fig. 27) denote the thickness of the object, a the angle made by the lever with the plane, and l the half length of the lever, then $d = l \sin a$. But the lever, in passing from the position AB to A′B′, moves through an

angle 2α, and the mirror also moves through 2α. Now, according to the laws of reflexion, the reflected ray moves

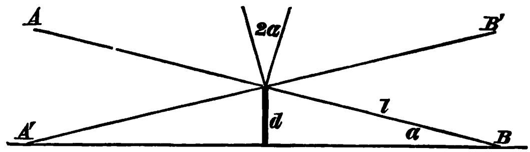


Fig. 27.

through an angle equal to twice that of the mirror, or 4α. If AT (Fig. 28) represent the scale at a distance L from

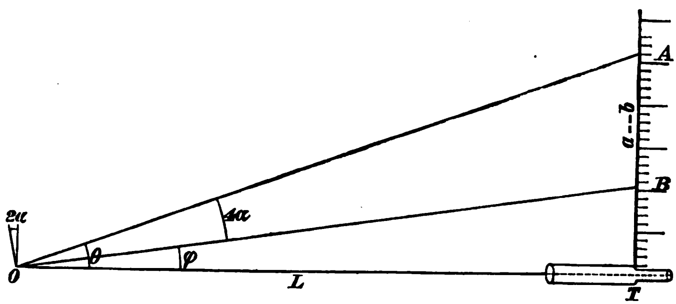


Fig. 28.

the mirror supposed to be at O, then $\text{AOT} - \text{BOT} = 4\alpha$. Calling these angles θ and ϕ, and the distances AT and BT, a and b respectively, we have

$$\tan 4\alpha = \tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{\dfrac{a-b}{\text{L}}}{1 + \dfrac{ab}{\text{L}^2}};$$

but, since ab is very small compared with L^2, the term involving it may be neglected, and hence

$$\tan 4\alpha = \frac{a-b}{\text{L}} = \frac{n}{\text{L}},$$

where n denotes the difference between the two readings.

Now, the angles being very small, $a = \sin a = \tan a$. Hence

$$d = l \sin a = \frac{ln}{4L}.$$

By increasing the distance between the lever and scale, by increasing the magnifying power of the telescope, and by diminishing the length of the lever, the sensibility of the apparatus may become so great that distances as small as a wave-length of light may be measured.

LESSON XVII.—Poggendorff's Method.

49. The method of using a mirror and scale, generally known as Poggendorff's method, may be applied in two ways. We may either employ a telescope to catch the reflected image of the scale, as in last lesson, or a beam of light from a luminous slit or other object may be made to impinge upon the mirror, the optical image of the object after reflexion being caught upon a divided scale. The first method is known as the *subjective* method, and is generally used in Germany; whilst the second, or *objective* method, is much used in this country in connexion with reflecting galvanometers and other similar instruments (see Vol. II.).

In either method the mirror and scale are so adjusted that in the zero position they are parallel to each other. If ACB (Fig. 29) be the scale and MR the mirror, then, if the mirror turn through the angle a, the reflected ray A′O will make the angle $A'OC = 2a$ with the zero line OC. Let the distance A′C be n scale divisions, each division being one millimètre, and a millimètre being regarded, for the nonce, as unit of length. Then we shall have

$$\tan 2a = \frac{n}{L},$$

L representing the distance in millimètres between C and O. The exact value of the angle a may now be easily

found by means of the ordinary trigonometrical tables, with which the reader is assumed to be familiar; but, except for

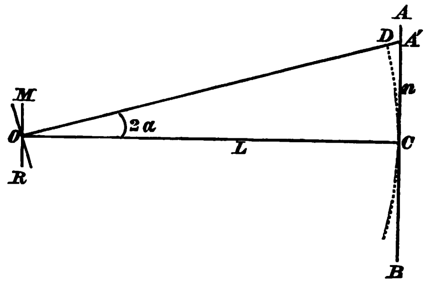


Fig. 29.

purposes of great accuracy, we may dispense with the use of such tables in the following manner.

As a rule, the angle through which the beam of light turns is very small, and may be taken as equal to its tangent, or, in other words, the tangent A′C (Fig. 29) may be taken as equal to the arc DC. We have thus the following proportion. If an arc equal to πL give 180° angular measure, what angular measure will an arc DC give, from which we find

$$2a = \frac{\mathrm{DC} \times 180^\circ}{\pi \mathrm{L}} = \frac{\mathrm{A'C} \times 180^\circ}{\pi \mathrm{L}} = \frac{n \times 180^\circ}{\pi \mathrm{L}},$$

or approximately—

$$a = \frac{n \times 103132''}{\mathrm{L}}.$$

The error made by using this approximation is usually very small. Thus, suppose that a millimètre scale is placed *only* one mètre from the mirror, and that $n = 100$, then

$$a = \frac{100 \times 103132''}{1000} = 2^\circ\ 51'\ 53{\cdot}2'',$$

whereas, calculated from the formula tan $2a = \frac{1}{10}$, we have

$$a = 2^\circ\ 51'\ 19''.$$

50. Frequently it is not *absolute* measurements but *comparative* measurements that are required. Thus, suppose that in the above experiment a deflection of 100 divisions, and in another one of 200 divisions, is obtained: let us find what error we commit by taking the ratio of the angles as 1 : 2. Calling the true angles α and α', we obtain from the correct formula $\alpha = 2^\circ\ 51'\ 19{\cdot}1''$ and $\alpha' = 5^\circ\ 39'\ 17{\cdot}8''$, the true ratio being 1 : 1·9805; the difference is thus not very great. The reductions, when only relative measures are required, will be avoided if the scale, instead of being a straight line, be the arc of a circle having the mirror for its centre. Sir William Thomson has adopted this plan in the scale provided for his Quadrant Electrometer.

51. Very frequently, instead of the actual angle being required, it is its tangent. For relative measurements with the straight scale the tangents are often taken as proportional to the direct deflections read off on the scale. Thus, in the comparison of two currents with a reflecting galvanometer, if deflections 100 and 200 be obtained, the currents (represented by the tangents of the real angles) are assumed to be as 1 : 2. If the scale be as before, at one mètre distance from the mirror, the divisions being millimètres, the actual ratio of the tangents of α will be as 1 : 1·9853.

52. Where greater accuracy is required, we make use of a better approximate formula obtained by taking the first two terms in the development of the angle or its tangent. We thus obtain

$$\alpha = 28^\circ{\cdot}648 \times \frac{n}{L}\left(1 - \frac{n^2}{3L^2}\right)$$

$$\tan \alpha = \frac{n}{2L}\left(1 - \frac{n^2}{4L^2}\right).$$

These formulæ are very convenient for comparative measure-

ments. Applying the first to the example selected, we have

$$\frac{a}{a'}=\frac{n\left(1-\frac{n^2}{3L^2}\right)}{n'\left(1-\frac{n'^2}{3L^2}\right)}=\frac{100-\ \cdot 33}{200-2\cdot 66}=\frac{1}{1\cdot 9799},$$

—a result, we see, almost identical with the correct one.

LESSON XVIII.—The Spirit-Level.

53. *Exercise.*—To adjust a level, to find the value of a division of its scale in angular measure, and to apply the instrument to the measurement of small angles.

Apparatus.—A spirit-level, a Whitworth plane surface, a level-tester, or apparatus to be used as a substitute for it.

The spirit-level is chiefly used for testing the horizontality of planes; but it is also applicable to the measurement of small angles differing slightly from the horizontal. It consists of a glass tube closed at both ends, and nearly filled with alcohol. The tube is not strictly cylindrical, but curved, so as to form the arc of a circle of large radius. It is graduated, and mounted in a metal case in such a way that the ends of the level may be raised or lowered with respect to the case by an adjusting screw (see *a*, Fig. 30). The *common* level has a flat base, the *striding* level (Fig. 30) has

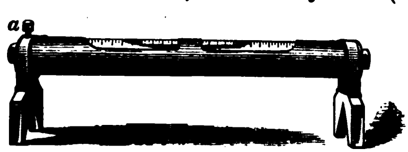


Fig. 30.—THE STRIDING LEVEL.

two legs provided with V's, so that it may be placed across the cylindrical pivots of a theodolite or transit telescope. Another form is the *hanging* level, adapted to hang from

the pivots by its two V's, which, in relation to those of the striding level, are in an inverted position.

Method.—When an accurate level is placed upon a horizontal plane the two ends of the air-bubble should be equidistant from the zero mark, which should correspond to the centre of the bubble. If it were quite easy to procure a strictly horizontal plane, it would be quite easy to adjust a level; the point is how to adjust it on a plane which may not be horizontal. The method of doing this will be best seen by a numerical illustration.

Let us first suppose that we have a *strictly adjusted level*, the extremities of whose bubble are at division 3 on each side of the zero, when the instrument stands on a plane A, which is horizontal.

Now, suppose we put it on a plane B, which is not perfectly horizontal, so that in one position the one extremity of the bubble stands at division 4. If we now reverse the level, turning it round through 180°, it is clear, from the principles of symmetry, that the other extremity of the bubble will now stand at division 4; so that the reading of, say, the right-hand extremity of the bubble will be the same in both positions.

Next, let us suppose that our level is *not accurately adjusted*, but that on the horizontal plane A one extremity of the bubble reads 4 and the other 2. Let it now be transferred to the plane B; it will be found that in the one position the right-hand extremity of the bubble will now read 5, while in the other position it will read 3. What we have to do is clearly to adjust the level by means of the screw, so that the right-hand extremity shall read 4 in both positions.

Thus the process in practice consists in placing the level on any plane, then taking the reading of both ends of the bubble, next reversing and repeating the readings. Then find the mean of the readings, direct and reversed, and adjust until the bubble rests at this mean position.

The process must be repeated until the readings are the same in whatever position the level stands. With a sensitive level it will be found extremely difficult to obtain perfection in this adjustment, and in this case it will be better not to attempt complete adjustment, but to find the small residual error.

In order to determine the value of a scale division in angular measure, the level-tester may be made use of. This consists of a substantial horizontal bar, with a second bar hinged to it and resting above it. The extremities of the bars most remote from the hinge are separated by a micrometer screw. The level is placed on the upper bar, and the position of the bubble is observed for different readings of the micrometer screw. Knowing the value of the micrometer divisions, and the distance between the screw and the hinge, the observations will afford us the means of determining the value of a scale division of the level in angular measure. A simpler method is to use wire of known diameter to raise one end of a bar, to which a level is attached, above a horizontal plane. If we are furnished with an instrument having a divided vertical circle, such as a theodolite or transit circle, the level may be attached directly to it, and the difference in reading required to bring the bubble from one position to another will give the angular value of a division of the level. Or, instead of trusting to the reading of the divided circle, we may make the horizontal cross-wire of the telescope coincide with a standard scale placed vertically at a known distance, and observe the difference in that reading of the scale which corresponds to the cross-wire, caused by moving the bubble of the level over one division. This latter observation may likewise be made by the telescope attached to the cathetometer, for in that instrument the telescope and the level move together.[1]

[1] The student is referred to works on Practical Astronomy for a fuller treatment of the level as used for angular measurement.

CHAPTER III.

Estimation of Mass.

54. MASS, or the quantity of matter in a substance, is estimated in terms of some standard. The legal standard

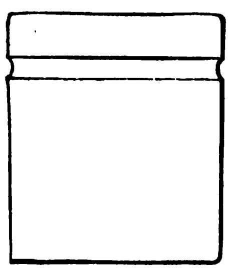


Fig. 31.

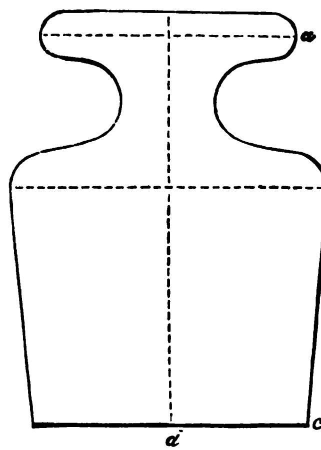


Fig. 32.

for Great Britain is a piece of platinum, of which Fig. 31 shows the exact shape and size. It is taken to represent 7000 grains, or 1 lb. avoirdupois.[1] The

[1] The construction of this standard is due to the late Prof. W. H. Miller, and the elaborate details of his experiments will be found in the *Philosophical Transactions* for 1856.

avoirdupois ounce is $\frac{1}{16}$ of a pound, and is equal to 437·5 grains.

The standard of mass in the metrical system is the **Kilogramme des Archives**, which is intended to have the same mass as a cubic décimètre of pure distilled water at its point of maximum density, reckoned at 4° C. The exact determinations of Kupffer have proved that the true mass of a cubic décimètre of water at 4° C. is 1·000013 kilogramme, so that, for practical purposes, the metrical standard may be taken to agree with the value which the founders of the system wished it to have. A careful copy (see Fig. 32) has been taken of the Kilogramme des Archives, which has been accepted in this country under the Act of 1864. The metrical system will almost exclusively be employed throughout these lessons. Its relation to such masses of the British system as are commonly used is seen in the following table :—

TABLE C.

RELATION OF BRITISH AND METRICAL STANDARDS OF MASS (after Dr. Warren de la Rue).

	English Grains.	Avoirdupois Pound = 7000 grains.
Milligramme . .	·015432	·0000022
Centigramme .	·154323	·0000220
Décigramme . .	1·543235	·0002205
Gramme . .	15·432349	·0022046
Kilogramme . .	15432·348800	2·2046213

1 grain = ·064798950 gramme.
1 lb. avoirdupois = ·45359265 kilogramme.
1 ounce avoirdupois = 437·5 grains = 28·349541 grammes.

To convert grammes into grains—

Log grammes + 1·188432 = log grains.

To convert grains into grammes—

Log grains + $\bar{2}$·811568 = log grammes.

APPROXIMATE VALUES.

Milligramme . .	·0154 grain ;	1 grain = 64·8 milligrammes.
Gramme . .	15·4 grains ;	28½ grammes = 1 avoirdupois ounce.
		454 grammes = 1 lb.

55. The **Balance** is used for the comparison of masses, and, for purposes of accuracy, it is invariably made with arms of equal length. The requirements of a delicate balance for laboratory work will be better appreciated by first considering the theoretical conditions that must be satisfied in order to make the instrument as sensitive and accurate as possible, and at the same time convenient for practical work.

LESSON XIX.—General Theory of the Balance.

56. By the sensibility of a balance we mean the angle through which it will turn for a given difference between the weights[1] in the pans. We shall now proceed to find the conditions which affect the sensibility of a balance. Let ABC

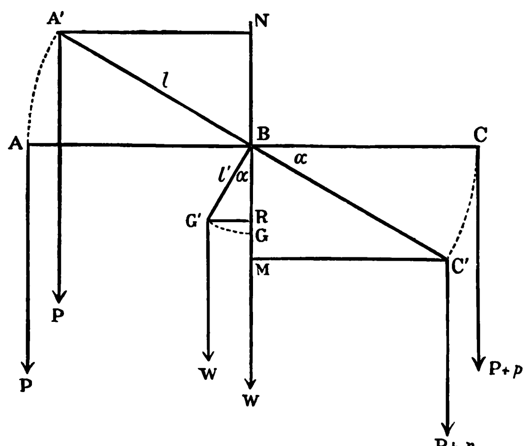


Fig. 33.

(Fig. 33) be the beam of the balance, its weight being W, and its arms each l units in length—the beam being fixed so as to

[1] For a discussion about weight and mass, see Appendix.

be movable about the point B. Now, the centre of gravity of the balance must be below this point, for if it were above B the equilibrium would be *unstable*, and the slightest addition of weight to either end of the beam would cause it to turn upside down; again, the centre of gravity must not be at B, for then the equilibrium would be *indifferent*, and the balance would rest equally well in any position. We shall therefore suppose the centre of gravity to be at G, at a distance l' from B. Now, if weights P and $P+p$ be suspended from A and C in such a manner as always to act vertically below their points of support, the balance will turn through a certain angle a, and, by the principle of the lever, the equation of equilibrium will be

$$(P+p).C'M = W.G'R + P.A'N$$
$$(P+p).l\cos a = W.l'\sin a + P.l\cos a$$
$$pl\cos a = W.l'\sin a,$$

or

$$\tan a = \frac{pl}{W.l'}.$$

Thus the sensibility, or the angle through which the balance turns for the given difference of weight, p, only depends on p, l, W and l', and we see from the formula that—

(1.) The longer the beam, the greater the sensibility.
(2.) The lighter the beam, the greater the sensibility.
(3.) The smaller the distance between the point of suspension of the balance and the centre of gravity, the greater the sensibility.

In the preceding demonstration we have assumed the point of suspension of the balance to be in the same line with the points from which the weights are suspended. If this were not the case, the sensibility would be found to depend likewise on the weight suspended from each arm; for let A and C be below B (Fig. 34), so that the arms each make an angle β with the horizontal line DE, the condition of equilibrium will now be seen to be—

$$P.A'N + W.G'R = (P+p).C'M$$
$$P.l\cos(\alpha-\beta) + W.l'\sin\alpha = (P+p).l\cos(\alpha+\beta)$$
$$P.l(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + W.l'\sin\alpha =$$
$$(P+p)l(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$$
$$\cos\alpha\,[P.l\cos\beta - (P+p).l\cos\beta] +$$
$$\sin\alpha\,[P.l\sin\beta + (P+p).l\sin\beta + W.l'] = 0.$$

Hence

$$\tan\alpha = \frac{p.l\cos\beta}{2P.l\sin\beta + p.l\sin\beta + W.l'} = \frac{1}{\frac{2P+p}{p}\tan\beta + \frac{W.l'}{pl}\sec\beta}.$$

In this case the sensibility will decrease with the load, since the greater the value of $2P+p$, the smaller will be the angle α.

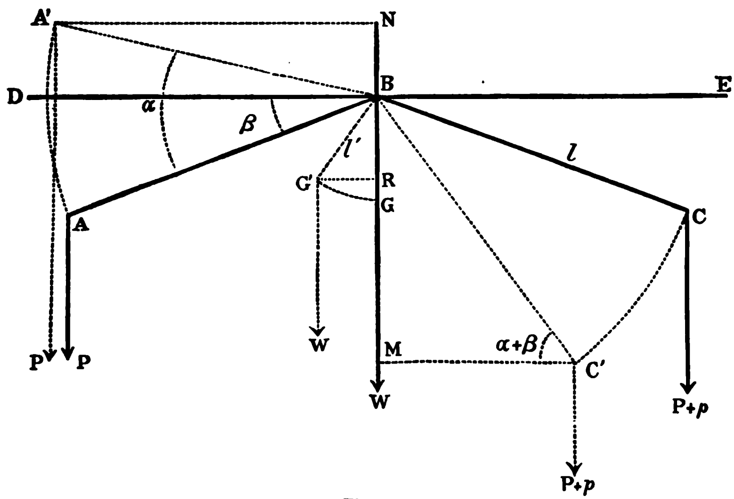


Fig 34

In the next place, let A and C be above B; the expression now becomes

$$\tan\alpha = \frac{1}{\frac{W.l'}{pl}\sec\beta - \frac{2P+p}{p}\tan\beta}.$$

The sensibility will, therefore, in this case increase with the load; but for that load which makes

$$\frac{W.l'}{pl}\sec\beta-\frac{2P+p}{p}\tan\beta=0$$

we should have $\tan\alpha=\infty$, or $\alpha=90°$, which means that the balance would be in unstable equilibrium.

Thus, if we are to have a balance whose sensibility is independent of load, and not liable to instability of equilibrium, we must observe the condition which will form the fourth requisite of a good balance, namely:—

(4.) The points of suspension of the scale-pans, and the point of suspension of the balance, should be in one line.

In the results now given the effect of friction has not been considered. Friction would decrease the sensibility by increasing the difficulty of turning, and also, when it occurred at the points of suspension of the weights, by preventing the weights from always acting vertically downwards from the same points, so that the balance would virtually have arms not of constant length. The fifth requisite is, therefore—

(5.) That at the points of suspension of the balance and of the weights there should be as little friction as possible.

It can be shown, by treating the balance as a compound pendulum, that the time of a small single oscillation of the balance is

$$t=\pi\sqrt{\frac{M.k^2+2Pl^2}{Ml'g}}$$

in which $\pi=3\cdot14159$, M denotes the mass of the balance-beam, k its radius of gyration, g the force of gravity, while P, l, l' have their previous signification.

Since much time would be consumed in weighing, if the time of vibration were excessive, we have here another requisite—

(6.) The time of vibration must not be too great.

In his endeavour to carry out these theoretical conditions the mechanician will find it necessary to make a compromise between them. For example, though he must make the arms long, the weight of the beam must be small, and yet there must be no loss of rigidity. And again, while the centre of gravity must be near the point of suspension, the time of vibration must not be too great.

We shall now describe a balance taken as a type of construction, which fulfils most of the requirements of physical laboratory work.

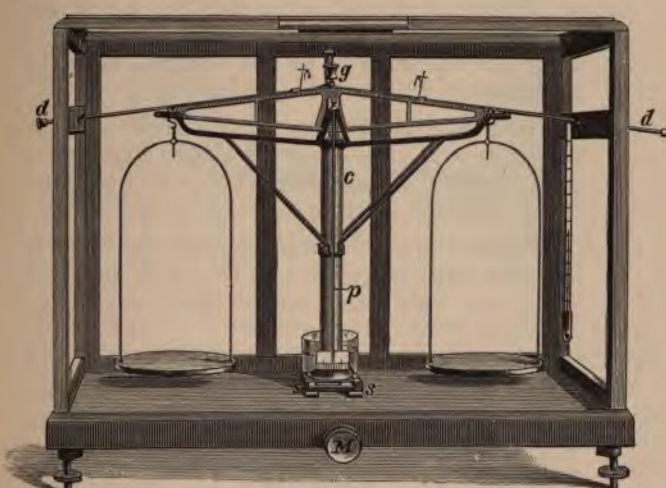


Fig. 35.—The Balance.

Lesson XX.—The Balance.

57. A general view of a 16-inch beam balance, made by

Oertling of London, capable of weighing from a kilogramme to half a milligramme, is given in Fig. 35. Its essential parts are—

(1.) The **Beam** is made of brass, in shape like an elongated lozenge, with cross arms, a form calculated to give rigidity combined with lightness. At K (Fig. 36), the central portion, there is a triangular brass prism with a knife-edge of agate turned downwards. At the ends of the beam there are similar knife-edges turned upwards (see *k*, Figs. 36 and 37). Above the central part is fixed a bob of brass, *g*, called the *gravity bob.* This may be raised or lowered by a slow screw motion. Below the gravity bob is a small vane, *v*, which, by turning it slightly towards right or left, may be made to correct small deviations from equilibrium.

(2.) The **Pointer**, *p*, is 321 mm. long, and is attached to the centre of the beam. It moves over a graduated *scale*, made of ivory, having twenty divisions, the spaces indicated by these divisions being 1·28 mm.

(3.) The **Pillar** P (Fig. 36), which supports the beam, is a hollow brass cylinder. At its top is an agate plane, on which the central knife-edge may rest.

(4.) The **Pan-Supports** (Figs. 36 and 37) consist each of a bent arm attached to a brass bar, which bears on its lower surface an agate plane, *a*, resting on the terminal knife-edge *k*. Each **Pan** is attached by a hook, H, to the pan-supports.

(5.) The **Knife-edges** and **Planes**.—By the use of these the balance-maker endeavours to reduce friction to a minimum. This involves the necessity of preserving the sharpness of the edges and the smoothness of the planes, and this in its turn

involves the condition that the edges and planes shall only be brought into contact when the balance is in use. Accordingly in all delicate balances there is a framework which brings these edges and planes into their required positions when the balance is about to be used. The constancy of this position is secured in a manner which will now be described.

(6.) The **Arrestment** is denoted by the shaded framework. It is attached to a central tube, *c*, which

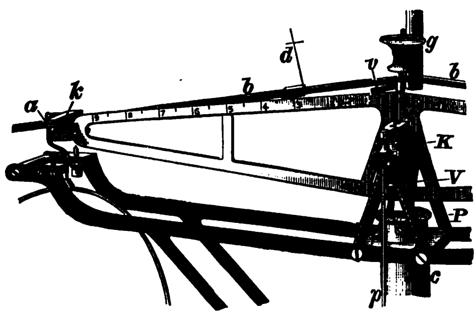


Fig. 36.—Balance Beam.

forms an outer covering surrounding the greater part of the pillar—the pillar itself being fixed. By means of the large milled head, M, acting outside the balance-case by an excentric movement, the arrestment may be raised or lowered. When the arrestment is at its highest position the central knife-edge is just lifted off the agate planes, and the terminal planes are also raised from the terminal knife-edges. Each pan-support has a hole, *h*, and slot, *s* (Fig. 37), into which, in the raised position of the arrestment, two screws, *r* and *r′*, with conical points attached to the end of the arrestment (Fig.

37), fit, just separating the planes from the knife-edges. At the same time two V-shaped portions of the arrestment lift the central knife-edge from its plane V (Fig. 36). As much depends on the perfect working of the arrestment, the mechanician endeavours to make the movement as smooth as possible.

(7.) Two arms, *d*, *d*, movable from the outside, can slide along two bars, *b*, *b* (Figs. 35 and 36), fixed above the

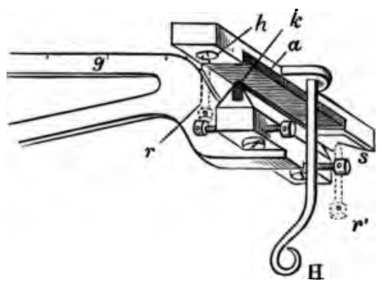


Fig. 37.—END OF BEAM.

balance beam, and thus enable a small weight of bent wire, called a **Rider**, to be placed on any of the graduated positions on the balance arms. The rider generally weighs one centigramme, and there are nineteen graduated positions on the balance beam at regular distances from the fulcrum, so that by this means differences in weight denoting $\frac{1}{20}$ of a centigramme or half a milligramme may be easily estimated.

The balance has a case with glass doors, so that while there is convenience of access there is at the same time freedom from currents of air. The instrument is supported on four levelling screws,

and is furnished with two spirit-levels, *ss* (Fig. 35), at right angles to each other.

57a. *Short Beam Balances.*—Especially when the body happens to weigh not more than 100 grms., there are many advantages in using a balance of much less massive construction than the one we have just described. The beam of the balance may then be made as light as possible, consistently with the requisite rigidity,[1] but if it is attempted to secure lightness by diminishing the length, we are brought into collision with the condition that the longer the beam the greater the sensibility. A choice or a compromise becomes necessary. Professor Dittmar, of Glasgow, on the one hand advocates long beams, while Bunge, of Hamburg, on the other hand, advocates short beams. Following out the latter plan, Bunge has constructed balances of high sensibility, in which the length of beam is extremely small, as will be seen from the following details:—

Maximum Weight.	Value of Scale Division.	Length of Beam.
200 grms.	·1 m. grm.	130 mm.
500 ,,	·1 or ·2 ,,	170 ,,
2000 ,,	·2 or ·5 ,,	240 ,,

The short beam type of construction is equally applicable to balances that are required to take heavy loads, the length of a Bunge balance adapted to take 50 kilogrammes being but 600 mm. One great advantage of this plan is that the time of vibration is reduced very considerably, so that weighings may be made with a rapidity impossible with the long beam balances.

58. *Position and Preservation of the Balance.*—The balance should be fixed on a firm table or stone slab in a position not exposed to the direct rays of the sun,

[1] Aluminum bronze is sometimes used on account of its lightness. The older balances of Fortin were made of tempered steel, a material admirable for its rigidity, but inadmissible on account of its liability to become magnetic.

and not liable to be affected by any marked inequality of temperature or by vibrations. Where gas lights are used, their position is of importance. The effect of a gas flame a few feet from a delicate balance is quite sufficient to destroy the accuracy of the weighings. The position of the balance having been chosen, it may be levelled, and then should be disturbed as little as possible. In order to keep the atmosphere dry, a vessel containing chloride of calcium, quicklime, or other substance for absorbing moisture, should be placed inside the balance-case and renewed from time to time. This is especially necessary when the knife-edges are of steel. No adjustments should be made until considerable experience in the use of the instrument has been attained. The knife-edges and planes will require to be cleaned occasionally. For this purpose the pan-supports may be removed, and the edges and planes dusted with a camel-hair pencil.

59. *Accuracy attained in the Construction of the Balance.*—Mechanicians have succeeded in reaching a very high standard of perfection in the manufacture of balances. The arms are made so nearly equal that careful measurements are necessary to find out their difference in length, which may not amount to more than one part in 50,000. The knife-edges are so keen that their width may not exceed $\frac{1}{200,000}$[1] of an inch, while the centre of gravity may be distant from the point of suspension by not more than $\frac{1}{1000}$ of an inch. With ordinary balances one milligramme may be detected in a load of one kilogramme or one part in a million. With balances such as those with short beams, used for weighings not exceeding 100 or 200 grammes, the hundredth of a milligramme may with certainty be detected.[2]

[1] W. S. Jevons, article "Balance," Watt's *Dictionary*.

[2] For a description of the balances of most modern construction, see the *Travaux et Mémoires du Bureau international des Poids et Mesures*.

The difference denoted by one division of the balance used by Professor W. H. Miller, in his estimation of the standard pound, was ·005 grain when two pounds were in each pan, and, as he estimated to hundredths of a division, he could thus detect a difference of not more than ·003 milligramme.

Fig. 38.—Box of Weights.

60. Weights.—The weights used are arranged in a box (Fig. 38) in the following order:—

Brass weights.			Platinum weights.	Platinum or aluminum weights.
1000 grms.	50 grms.	5 grms.	0·5 grm.	0·05 grm.
500	20	2	0·2	0·02
200	10	2	0·2	0·01
100	10	1	0·1	0·01
100				

The weights ·005, ·002, ·001, being very small, are seldom used. A wire of aluminum or gilt brass (Fig. 39), called a centigramme rider, which may be placed at different points along the beam in the manner just described, is much more convenient.

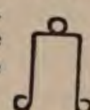


Fig. 39.
The Rider.

The smaller weights are protected when not in use by a slip of glass placed over the compartments that contain them. The larger weights should always be handled by means of the forked piece

of ivory, and the smaller weights by pincers (Fig. 38). They should be covered with some metal which is not subject to oxidation. They are often platinised. If they are gilt, care must be taken that they do not come in contact with mercury. A similar remark applies to the scale-pans.

LESSON XXI.—Method of Using the Balance.

61. *Apparatus.*—The balance, which we shall suppose to have been put into good adjustment, a box of weights, a rider, a camel-hair brush, and an object to be weighed.

Method.—On the left-hand pan, which may be named the object-pan, place the body to be weighed, and in the middle of the right-hand or weight-pan such weights as are estimated sufficient to counterbalance the body. Suppose that $50 + 20$ grms. are used, and that on partially lowering the arrestment the pointer is seen to move towards the object, indicating that these weights are too great. Suppose next that on substituting 10 for 20 grms. the weight is found to be too little. The subsequent steps of the process are as follows:—

$50 + 10 + 5$, too great.
$50 + 10 + 2$, too little.
$50 + 10 + 2 + 2$, too little.
$50 + 10 + 2 + 2 + \cdot 5$, too little.
$50 + 10 + 2 + 2 + \cdot 5 + \cdot 2$, too little.
$50 + 10 + 2 + 2 + \cdot 5 + \cdot 2 + \cdot 1$, still too little, but not far from the truth. ·005 is next added, which proves too great; and finally, on substituting ·0025 for it, the pointer swings equally on both sides of the central line of the scale. The weight is thus found to be 64·8025 grms.

As the student becomes familiar with the balance he will learn to weigh quickly and know from the swinging of the balance how much to add in order to obtain equilibrium.

62. During the process of weighing it will be necessary to observe several precautions. For the sake of convenience let us arrange under one head the general course of procedure in using the balance, as well as the special precautions necessary.

Precautions in Weighing.

1. See that the rider is in its place—on its supporting arm, and not likely to touch the beam during the oscillations of the balance.
2. Brush the pans with a flat camel-hair brush.
3. Lower the arrestment to see whether the balance swings equally on both sides of the scale. If not, adjust the vane carefully until it does so.[1]
4. Do not stop the swinging of the balance with a jerk. It is best to stop it when the pointer is at its central position.
5. Stop the swinging of the balance when weights are to be added or taken away.
6. The position of the observer should be central, so that there may be no parallax in observing the position of the pointer.
7. If the balance is nearly in equilibrium there may be a difficulty in getting up a vibration; in this case gently waft the air over one of the pans. Or the arrestment may be raised and lowered again; one or two attempts will set up the required swinging.
8. Place the large weights in the centre of the pan and the smaller weights in the order of their denomination.

[1] The frequent adjustment of the vane is objectionable, tending to produce injury of the balance; so that when a balance is in much use it is better to correct for slight want of balance by adding weight to one arm, or by making an allowance in scale divisions for the deviation of the pointer from the centre.

9. Final weighings must be made with the balance-case closed, and care must be taken that the pans do not swing.
10. Do not weigh a body when hot; the heat causes air-currents, which affect the weighing.
11. All substances liable to injure the pans must be weighed in appropriate vessels.
12. Hygroscopic bodies must be weighed in closed vessels, as also volatile liquids.
13. Remove the weights from the pans, the rider from the beam, and close the balance when the weighing is finished.

Lesson XXII.—Weighing by Vibrations.

63. *Exercise.*—To manufacture and then to weigh exactly a centigramme rider.

Apparatus.—Some aluminum wire (No. 28 B. W. G., *i.e.* ·014 inch = ·35 mm. in diameter, see Table B_1) and a millimètre scale.

Method.—Weigh a piece of wire to within ·5 of a milligramme, straighten the wire and measure its length. Calculate from this what would be the length of ·01 grm. of the wire, and cut off a piece rather longer than you have calculated. For example, suppose that 325 mm. weighs ·0875 grm., ·01 grm. will therefore have a length = $325 \times \frac{\cdot 01}{\cdot 0875} = 37\cdot 1$ mm. A piece 37·5 mm. in length must therefore be cut off and weighed in the following manner—known as the method of vibrations:—

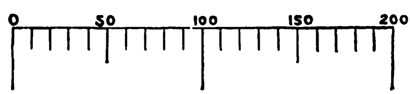


Fig. 40.

Suppose the ivory scale of the balance (Fig. 40) to be

numbered from left to right, the extreme left being counted 0, the middle 100, and the extreme right 200. Set the balance vibrating with empty pans over about 4 of the ivory divisions, and after the pointer has made one or two swings, so as to give it a regular motion, note down its exact turning points, estimating by the eye (aided, if necessary, by a large lens) to one-tenth of a division, taking care at the same time to avoid parallax. Suppose the following observations have been made at regular intervals of time:—

Left.	Right.
(1) = 88	...
...	(2) = 120
(3) = 92	...
...	(4) = 116
(5) = 95	...
...	(6) = 112
(7) = 99	

The mean of the four left is 93·5, and of the three right 116. Hence the true position of rest will be $\frac{93{\cdot}5 + 116}{2} = 104{\cdot}75$. The following explanation will render evident why this method of procedure is adopted.

In estimating what will be the final position of the pointer of a balance while it is still in a state of oscillation, the effects of friction and atmospheric resistance in continuously shortening the swing must not be disregarded. Suppose, for instance, that, as above, the pointer when first observed at the extreme left of its swing indicates the scale number 88, and then at the extreme right of its next swing the scale number 120. Now it will not be correct to take the mean of these, or 104·00, as denoting the ultimate position of rest of the pointer. For, since the oscillations are always becoming smaller, the first of these two will be farther to the left of the true point than the second will be to the right of it. We may, however, compare together a series of left-hand and a series

of right-hand numbers, provided the mean time of occurrence of the one set be precisely the same as that of the other, for in this case we are justified in imagining that friction, etc., will have acted equally on both sets. Thus, for instance, referring to the series given above, we may use the four left-hand and the three right-hand numbers, and obtain the final position 104·75.

Or we may take (1) (3) and (5) and compare them with (2) and (4), in which case we obtain as the final position 104·835; or we may compare (3) (5) and (7) with (4) and (6), when we shall obtain 104·67. Or again (1) and (3) with (2), when we shall get 105·00; or (3) and (5) with (4), when we shall get 104·75; or finally, (5) and (7) with (6), when we shall get 104·50. Probably the mean derived from the whole body of observation, or 104·75, is the most accurate, but none of the above means differ from this so much as that which we should have obtained had we used two observations such as (1) and (2), for which the time-epochs were different.

We next proceed to find the sensibility of the balance. This has already been defined as the angle through which the balance will turn for a given difference of load. A milligramme is generally taken as the unit of difference, and the angle, being small, is measured by the divisions of the scale. Let us now, therefore, add a milligramme to one side of the balance and again determine its resting point. Two observations give 89·84 and 89·80, the mean being 89·82. The excess of weight has therefore caused a difference of $104{\cdot}75 - 89{\cdot}82 = 14{\cdot}93$, or 15 divisions nearly, so that the sensibility of the balance may be called 15.

Now place the short piece of aluminum wire in the left-hand pan, and the ·01 grm. weight in the other, and again determine the zero point. Let us suppose this to be 106·35, denoting a difference of $106{\cdot}35 - 104{\cdot}75 = 1{\cdot}60$. The wire is thus found to weigh $\frac{1{\cdot}6}{15} = 0{\cdot}11$ of a milligramme

too much, or its real weight is ·01011. This will be sufficiently accurate for some purposes, but it may be readily brought more nearly in accord with the required value by scraping off a small quantity of metal and again weighing. The wire, when its weight is finally adjusted, may be bent by a small forceps into a form convenient for use as a rider.

64. *Advantages of the Method of Vibration.*—This method should always be preferred to the method described in the previous lesson for the following reasons :—

(1.) It is not necessary that the pointer should be adjusted so as to point accurately to the middle of the scale.

(2.) The method by vibrations takes account of the decay of the oscillations due to friction, and the resistance of the air.

(3.) "It is better as a rule to make a measurement depend not on a trial as to whether two quantities are equal, for equality is never perfect, but rather on a trial as to how much they differ."—Kohlrausch—*Physical Measurements.*

Lesson XXIII.—The working conditions of the Balance.

65. The instrument having been levelled, and the vane adjusted, we proceed to test the accuracy of the adjustments which the maker has made, and to determine generally the working conditions.

(*a*) *The Time of Vibration.*—Set the balance vibrating, and note the time which elapses between 10 passages of the pointer over the centre of the scale.

Example—

		Minutes.	Seconds.	
Passage	0	2	53	Difference = 149 seconds.
,,	10	5	22	

A single vibration is thus made in 14·9 seconds.

The time of vibration gives us a delicate means of detecting any variation in the conditions of the balance. For a given balance and a given load the greater the time of vibration the greater the sensibility. The time of vibration varies with the length of the beam and with the load.

Example—

	Bunge 5-inch Beam.	Oertling 16-inch Beam.[1]
Pans unloaded	7·5 seconds.	20 seconds.
Load 50 grms.	9·5 ,,	28 ,,
Load 100 grms.	11·4 ,,	31 ,,

From 12 to 15 seconds should be the maximum for ordinary weighings.

(β) *Constancy of Zero Point.* — On account of slight variations in the relative positions of the planes and edges, on account also of disturbances due to the inequality of expansion, or to the flexure of the beam, the zero point may vary somewhat from time to time. In delicate experiments, as, for instance, the comparison of a set of weights, these variations may be so great as to cloak the results. The balance should, therefore, be tested repeatedly, in order that a knowledge of the extent of its variations may be acquired.

Example.—The knife-edges were dusted and the following results obtained:—

I. The balance set swinging and allowed to come to rest. Zero point, 96·00.

II. Zero point determined during motion four times.

Swings 56 to 137	Zero point 95·50
,, 66 to 123	,, 95·57
,, 82 to 110	,, 95·65
,, 63 to 132	,, 95·70.

III. Load of 200 grms. placed in each pan and balance allowed to swing a little time — the weights re-

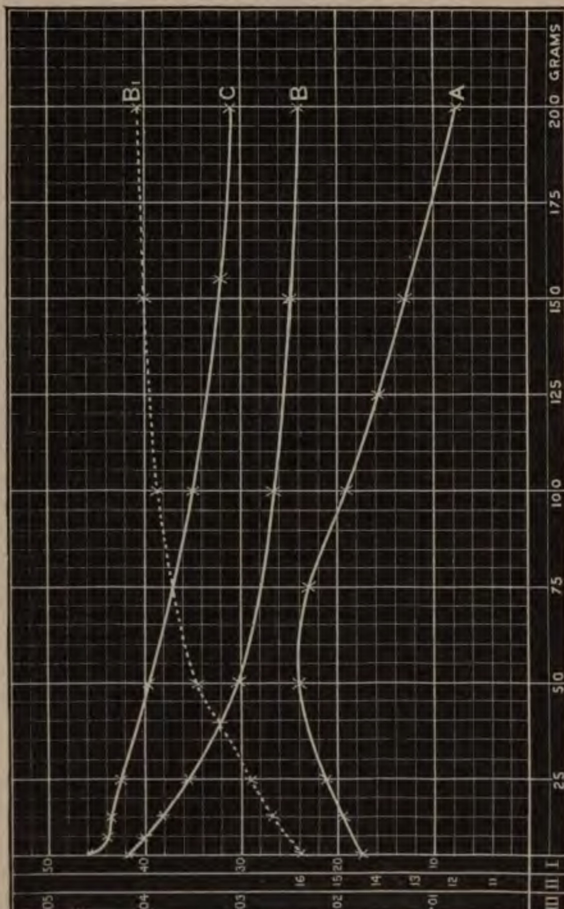
[1] This balance had its gravity bob raised above the normal position.

moved and zero point again determined. There was obtained (1) 97·0, (2) 96·3

IV. Load of 500 grms. in each pan—treatment as above. Zero point at first 99·0; after a little time it returned to its original zero point.

V. Load of 1000 grms. Zero point at first 99·2, afterwards 95·5.

(γ) *The Sensibility of the Balance.*—Determine the sensibility with empty pans, and also with various loads, for instance 5, 10, 15, 20, 25, 50, 100, 150, 200, 300 grms., up to half the maximum load for each pan. Next plot a curve on paper ruled with squares (see Appendix), with loads as abscissæ, and the corresponding sensibilities as ordinates. A convenient scale should be chosen, so that errors of observation may not be unduly magnified. In Fig. 41, A is such a curve formed from an Oertling 16-inch balance, and from it we see at once that there is an increase of sensibility up to 55 grms., after which for greater loads the sensibility decreases. The probable explanation of this behaviour is that for empty pans the terminal knife-edges are slightly above the central one, but, as the beam bends more and more with each additional load, a certain load will be reached which will bring the three knife-edges to the same line. After this, greater weights continuing to lower the terminal knife-edges will thus diminish the sensibility. B of Fig. 41 shows another such curve obtained from a Bunge short-beam balance. Here from the first there is a decrease of sensibility. C of Fig. 41 shows the curve obtained from another type of a short-beam balance; here the decrease at first is more rapid. The information supplied by these graphical representations may be utilised in weigh-



I. Sensibility in scale divisions for B and C.
II. Sensibility in scale divisions for A.
III. Milligramme values for a scale division for B_1.

Fig. 41.—Curves of Sensibility of Balance.

ing. Thus suppose that during a weighing we know that the pointer is displaced 8 divisions from its position of equilibrium in the right-hand direction, and suppose that the load is 120 grammes, the curve B of Fig. 41 tells us that the sensibility for this load—that is to say, the displacement of the pointer for 1 milligramme of excess—is 25·6. Hence $\frac{8}{25 \cdot 6} = \cdot 31$ of a milligramme is the amount[1] which must be added to the weights in order that they may counterbalance the body.

It will, however, be more convenient if we plot a second curve giving the excess of weight in milligrammes, which corresponds to one division of the scale. The dotted line B_1 (Fig. 41) shows such a curve for a Bunge balance. Here the abscissæ are the same as before, but the ordinates, instead of representing the sensibilities as in B, represent their reciprocals. To apply this curve to the example just given, we see that with a load of 120 grammes a deviation of 8 divisions would represent $8 \times \cdot 0392 = \cdot 3136$ milligrammes. Such a curve will greatly facilitate weighing by the method of vibrations, for, remembering that as a rule we cannot be sure to within ·1 milligramme of our weighing, it will be sufficient to take two decimal places, so that the multiplication may be performed mentally. As there are various circumstances which affect the sensibility, the curve should be redetermined from time to time. But with ordinary care the sensibility will be found to be fairly constant.

(δ) *Ratio of the Balance Arms.*—Ordinary weighings are made on the assumption that the balance arms are of equal length. As it is impossible for the

[1] The divisions of the scale increasing from left to right, and the body to be weighed being always placed in the left-hand pan, increased readings will always denote increased weight of body and *vice versâ*.

maker to secure a perfect equality of this nature, it is important to find out how far he has succeeded.

For this purpose place two weights of the same nominal value, P and P′, one in each pan, and obtain equilibrium with reference to the previously observed zero point; next interchange the positions of the weights, and again balance. Suppose that when P is placed in the left-hand pan p must be added to it, and that when it is placed in the right-hand pan p' must be added to it, in order to procure equilibrium. Calling R and L the length of the right and left arms respectively, we obtain

$$\begin{aligned} L(P+p) &= P'R \\ LP' &= (P+p')R\,; \end{aligned}$$

wherefore

$$L^2P'(P+p) = P'(P+p')R^2,$$

or

$$\frac{L}{R} = \left(\frac{P+p'}{P+p}\right)^{\frac{1}{2}} = \left(\frac{1+\frac{p'}{P}}{1+\frac{p}{P}}\right)^{\frac{1}{2}} = 1 + \frac{p'-p}{2P} \text{ nearly.}$$

To p and p' the proper algebraical sign must be given.

Example—

Left arm.	Right arm.
P = 200 grms.	P′ + ·008073.
P′ + ·0005	P = 200.

Hence

$$\frac{L}{R} = 1 + \frac{\cdot008073 - \cdot0005}{400} = 1\cdot00001893.$$

A second determination with 300 grms. gave

$$\frac{L}{R} = 1\cdot0000192.$$

(ε) *The Actual Sensibility*, or the angle of deviation for an excess of one milligramme, may be found from the observed sensibility, if we know the length of the pointer and of a scale division. In our Oertling balance now in use one small division of the scale

is equal to $1'\cdot37$. Hence, if the observed sensibility were 15, this would correspond to an angle of $20'\cdot5$.

(ζ) *The Gravity Bob.*—We have supposed the preceding observations to have been made with the gravity bob in a fixed position. The maker will probably have placed it in the best position for the balance, and it will very seldom be necessary to disturb it.

The gravity bob provides us with a means of regulating the sensibility of the balance, for by its means the distance of the centre of gravity from the centre of suspension may be changed. When screwed up, the centre of gravity is brought nearer to the centre of suspension; the time of vibration is thus lengthened and the sensibility increased. When the gravity bob is too high, a position of instability will be reached either for empty pans or for a certain load. In the latter case we see from the formula (Art. 56) that the terminal knife-edges must have been above the central one, and that the sensibility will therefore gradually increase with the load until at length a position of instability is reached.

The following is an instance of this state of things obtained from a balance in which the gravity bob had been disturbed:—

Load.	Sensibility
0	10·05
10	10·10
20	10·65
40	10·75
80	10·85
120	unstable.

(η) *Constancy of Weighings.*—The final test of the constancy of a balance consists in weighing a body several times in succession with the greatest ex-

[redacted]

actitude. Corrections must be made for variation in the atmospheric density, as explained in the chapter on "Density." The following is the result of the weighing of a platinum crucible on different days.[1] The weighings were made on a balance by Shickert of Dresden.

	Without air correction.	With air correction.
	Grms.	Grms.
1.	93·97028	93·96974
2.	93·97035	93·96973
3.	93·97004	93·96971

66. The fatigue of balance operations may be diminished, and the accuracy of the results increased, by using a telescope of low magnifying power, placed a few feet from the instrument. In this case it will be necessary to use some means of illuminating the scale, for which purpose a bull's-eye lantern will be found convenient. The use of a telescope will also eliminate any error due to parallax.[2]

Lesson XXIV.—Exact Weighing.

67. In exact weighings there are three principal sources of error to be considered:—

(1) Inequality of the arms of the balance.
(2) Difference of density between the body weighed and the weights.
(3) Errors of the weights.

[1] This example has been furnished by Professor Humpidge of the University College of South Wales.

[2] This source of error might also be eliminated by using mirror glass for the scale, and taking care that the pointer always coincided with its image.

The effect of *inequality of the arms of the balance* may be eliminated by two methods, known as those of Borda and of Gauss.[1]

In *the method of Borda* the body to be weighed is counterbalanced by weights, shot, etc., and finally adjusted to a position as near that of equilibrium as possible by means of fine sand, light paper, or other similar substances. The body is now removed and replaced by standard weights until equilibrium is again secured. The weight in the pan will now evidently denote the mass of the body weighed, inasmuch as the body and this weight, placed under the same circumstances, each exactly counterbalances the matter in the other pan.

In making use of this method, as it would occupy much time to adjust to exact equality, it is best to adopt the plan of vibrations.

Example.—A bottle was placed in the right pan and counterpoised. Zero point, 105·0.
The bottle was removed and replaced by 21·818 grms. Zero point, 102·1. Whilst 21·817 grms. gave 117·1.

The true weight is therefore

$$21{\cdot}817 + \frac{12{\cdot}1}{15} \text{ mgm.} = 21{\cdot}8178 \text{ grms.}$$

The *method of Gauss* consists simply in weighing first in one pan and then in the other. Thus, if a body of true weight W weighs A when placed in the right-hand pan and B when placed in the left-hand pan, then calling R and L the lengths of the respective arms, we have

$$WR = AL$$
$$WL = BR;$$

[1] A knowledge of the ratio of the arms will furnish us with a third method; since, however, this ratio may vary with the load on account of difference in rigidity between the two arms, this method is seldom employed.

therefore

$$W^2 = AB,$$

or

$$W = \sqrt{AB}.$$

Thus the true weight is the geometrical mean of the apparent weights. Since as a rule A and B are very nearly equal, we may, in most cases, use the arithmetical mean instead. Thus the arithmetical mean of 1·000 and 1·001 is 1·0005, whilst the geometrical mean is 1·0004998. It would be impossible to detect the difference between the two by the balance.

The method of Gauss has several advantages over that of Borda. It is more correct, for it can be shown that the probable error of Borda's method is decidedly greater than that of Gauss. Furthermore, less time is taken, and there is the additional advantage of the one weighing furnishing a check on the other.

As the weights increase, it becomes more important to employ one or other of these methods. We give an actual series of weighings made with an Oertling balance, showing this:—

Left Pan. Grms.	Right Pan. Grms.	Difference. Grms.	Arithmetical Mean. Grms.	Geometrical Mean. Grms.
907·840	907·813	·027	907·8265	907·8264
900·594	900·570	·024	900·5820	900·5819
254·884	254·878	·006	254·8810	254·8811
108·501	108·498	·003	108·4995	108·4995

Difference of Density between the Body weighed and the Weights.—The balance will only give accurate comparisons of mass when the weighing is performed *in vacuo*, unless the body weighed happens to have the same density as the standard weights. The reason is that if the body and the weights are of unequal volume, they will displace dif-

ferent amounts of air, and hence lose weight unequally. It is, however, easy to make the requisite correction. Let W be the true weight *in vacuo* of the body, which is balanced in air by P standard grammes (*i.e.* weights which *in vacuo* would accurately have this value). If Δ be the density of the body, and B that of the weights, the volume of the body will be $\frac{W}{\Delta}$, and that of the weights $\frac{P}{B}$.

If σ be the weight of unit volume of air at the time of weighing, then the body will lose in air $\frac{\sigma W}{\Delta}$, and the weights $\frac{\sigma P}{B}$; but in air there is equilibrium, so that

$$W - \frac{\sigma W}{\Delta} = P - \frac{\sigma P}{B}.$$

Hence

$$W = P\frac{1 - \frac{\sigma}{B}}{1 - \frac{\sigma}{\Delta}} = P + P\sigma\left(\frac{1}{\Delta} - \frac{1}{B}\right)$$

nearly, since σ is small in comparison with B and Δ.

Thus we see that when the density of the weights is greater than that of the body, something has to be added to the apparent weight in order to obtain the true weight. Let us take the gramme as our unit of weight and the cubic centimètre as our unit of volume; σ will then be the weight of one cubic centimètre of air which has a mean value of ·0012 grms., a result sufficiently accurate for nearly all purposes. If we likewise assume that we are using brass weights (almost universally used) with a mean density of 8·4, we may easily construct a table giving the value of the above correction for different values of Δ.

The following table has been compiled in this manner:—

TABLE D.

REDUCTION OF WEIGHING TO VACUO (σ = ·0012, B = 8·4).

Δ Density of Body weighed.	$\sigma\left(\frac{1}{\Delta}-\frac{1}{B}\right)$ Correction in Milligrammes per Gramme of Weight.	Example.
0·7	+ 1·57	Ether.
0·8	+ 1·36	Alcohol.
0·9	+ 1·19	Olive Oil.
1·0	+ **1·057**	**Water.**
1·1	+ 0·95	
1·2	+ 0·86	
1·3	+ 0·78	
1·4	+ 0·71	
1·5	+ 0·66	Nitric Acid.
1·6	+ 0·61	
1·7	+ 0·56	
1·8	+ 0·52	
1·9	+ 0·49	
2·0	+ 0·46	
2·5	+ **0·34**	**Glass.** [1]
3·0	+ 0·26	
4·0	+ 0·16	
5·0	+ 0·10	
6·0	+ 0·06	
7·0	+ 0·03	Iron.
8·0	+ 0·01	
8·4	0·00	Brass.
9·0	− 0·01	
10·0	− 0·02	
12·0	− 0·04	
13·6	− **0·0546**	**Mercury.**
14·0	− 0·06	
16·0	− 0·07	
18·0	− 0·08	
20·0	− 0·08	

The Errors of the Weights.—The exact adjustment of weights has been brought to a high state of perfection by

[1] Glass, such as specific gravity bottles are made of.

several makers, so that the errors of the weights furnished by them may, as a rule, be neglected.

In the next lesson we shall explain how these errors may be determined. This can, however, only be done by using a very good balance, and taking all the necessary precautions.

Lesson XXV.—Errors of Weights.

68. *Exercise.*—To compare a set of weights with a standard weight.

Method.—The problem consists in a series of comparisons by the method of Gauss, and will best be illustrated by an actual example. It was required to compare a set of weights by Staudinger of Giessen with a standard 100 gramme weight by Oertling (100_0). The comparison was made by means of a "Ladd and Oertling" Balance. In this balance the upper part of the pointer is furnished with a system of mirrors, by which a scale fixed at the end of the balance-case was reflected into a telescope. The scale in this particular balance was divided into 100 parts, the central division being 50. To this the zero point was adjusted.

Each operation consisted in three observations of the zero point, as recorded below:—

I.		II.	
100_0 Left, 100 Giessen $=100_\alpha$ Right.		100_α Left.	100_0 Right.
52·0	...	7·4	...
...	95·7	...	43·0
53·5	...	8·3	...
...	94·3	...	42·1
54·8		9·2	
53·43	95·00	8·3	42·55
Zero point $=74·21$.		Zero point $=25·42$.	

III. One milligramme was added, the weights remaining

in the pan, the zero point again determined, and the sensibility found to be 15·4.

The *difference* of the weights will in scale divisions be equal to half the difference of the change of zero point, or

$$\frac{74{\cdot}21 - 25{\cdot}42}{2} = 24{\cdot}395 \text{ divisions, or } \frac{24{\cdot}395}{15{\cdot}4} = 1{\cdot}584 \text{ mgrms.}$$

Since 100_o is the heavier, therefore the true weight of 100_α will be 99·99842. A second 100 grm. weight (Giessen), distinguished as 100_β, weighed against 100_o, as standard, gave 99·99851.

The next step was to weigh 100_β against $50 + 20 + 10 + 10 + 5 + 2 + 1 + 1 + 1$ (Giessen). The latter were found ·00034 too light. The results of subsequent weighings were as follows:—

50 against 20 + 10 + etc. . .	latter too light by ·00015
20 against $10_\alpha + 10_\beta$ . . .	latter too light by ·00023
10_α against 10_β	latter too light by ·00002
10_β against $5 + 2 + 1 + 1 + 1$.	latter too light by ·00002.

Let the true weight of

$$\begin{aligned} 100_\beta &= A \\ 50 &= B \\ 20 &= C \\ 10_\alpha &= D \\ 10_\beta &= E \\ (5+2+1+1+1)=10_\gamma &= F. \end{aligned}$$

From the preceding determinations we obtain

$$\begin{aligned} 100_o &= A + {\cdot}00149 \text{ or } A = 99{\cdot}99851 \\ A &= B + C + D + E + F + {\cdot}00034 \\ B &= C + D + E + F + {\cdot}00015 \\ C &= D + E + {\cdot}00023 \\ D &= E + {\cdot}00002 \\ E &= F + {\cdot}00002. \end{aligned}$$

Expressing each of these in terms of F, we have

$$\begin{aligned} E &= F + {\cdot}00002 \\ D &= F + {\cdot}00004 \\ C &= 2F + {\cdot}00029 \\ B &= 5F + {\cdot}00050 \\ A &= 10F + {\cdot}00119 = 99{\cdot}99851. \quad \text{Hence } F = 9{\cdot}999732. \end{aligned}$$

We then find immediately the following values:—

$$
\begin{aligned}
B &= 49{\cdot}999160\\
C &= 19{\cdot}999754\\
D &= 9{\cdot}999772\\
E &= 9{\cdot}999752\\
F &= 9{\cdot}999732.
\end{aligned}
$$

The sum of these, 99·998170, when added to the difference obtained from comparing A against these weights, namely, ·00034, should give the value of A. We see that this is the case, for 99·99817 + 00034 = 99·99851. This proves the working of the equations to be correct, but gives no indication of the accuracy of the individual results.

To control the weighings, check weighings were performed; for example, 100_α compared directly with 100_β gave a difference of ·0001.

The process was applied to the weights of smaller denomination in exactly the same way. The final values are given in the following table[1]:—

$100_\alpha = 99{\cdot}99842.$					$100_\beta = 99{\cdot}99851.$				
50	49·99916	5	4·99990	·5	·499995	·05	·04998	·005	·00500
20	19·99975	2	1·99992	·2	·19999	·02	·01994	·002	·00200
								Riders.	
10_α	9·99977	1_α	1·00002	$\cdot1_\alpha$	·10001	$\cdot01_\alpha$	·01001	$\cdot01_\alpha$	·00996
10_β	9·99975	1_β	·99998	$\cdot1_\beta$	·09995	$\cdot01_\beta$	·00996	$\cdot01_\beta$	·010025
10_γ	9·99973	1_γ	·99995	$\cdot1_\gamma$	·09989			$\cdot01_\gamma$	·01002

In this table 10_γ, 1_γ, $\cdot1_\gamma$ are made up of small weights.

As a rule, it is the relation between the weights that

[1] This table would in practice be supplemented by information to enable us to distinguish weights of the same nominal value from each other. As a rule, slight scratches or differences in the engraving of the figures will serve the purpose.

is required to be correct rather than the precise absolute values of the individual weights. For this purpose the mean error of all the values given above is found, and that weight selected as normal whose error is near to the average error. In this case the 50 grm. weight is convenient, and accepting this as the standard of comparison, the following numbers are deduced :—

$100_\alpha = 100 \cdot 00010.$		$100_\beta = 100 \cdot 00019.$
$50 = 50 \cdot 00000$	$\cdot 5 = 4 \cdot 99998$	$\cdot 5 = \cdot 50000$
$20 = 20 \cdot 00009$	$2 = 1 \cdot 99995$	$\cdot 2 = \cdot 19999$
$10_\alpha = 9 \cdot 99994$	$1_\alpha = 1 \cdot 00004$	$\cdot 1_\alpha = \cdot 10001$
$10_\beta = 9 \cdot 99992$	$1_\beta = 1 \cdot 00000$	$\cdot 1\beta = \cdot 09995$
	$1_\gamma = \cdot 99997$	

CHAPTER IV.

Measurement of Area and Volume.

Area.

69. The units of area or surface extent are derived from those of length. The following table exhibits the relation between the metrical and the British systems of estimating areas :—

Table E.

Relation between Metrical and English Measures of Area.

	English square inches.	English square feet.	English square yards.
1 square millimètre	·00155	·0000107	·0000012
1 square centimètre	·155006	·0010764	·0001196
1 square décimètre	15·500591	·1076430	·0119603
1 square mètre .	1550·059099	10·7642993	1·1960333

1 square inch = 6·4513669 square centimètres.
1 square foot = 9·2899683 square décimètres.
1 square yard = ·83609715 square mètres.

70. The areas of figures of regular shape may be directly derived by the rules of mensuration from linear measurements. The following table gives the cases of most frequent occurrence :—

TABLE F.

MENSURATION OF AREAS.

Triangle	$=$ Base $\times \frac{1}{2}$ perpendicular.
Circle . .	$=\pi r^2$, where $r=$radius and $\pi=3{\cdot}14159$.
Parabola	$=$ Base $\times \frac{2}{3}$ height.
Ellipse	$=\frac{\pi}{4}ab$, where $a=$major, $b=$minor axis.
Whole surface of right cylinder	$=2\pi rl+2\pi r^2$, where $l=$length and $r=$radius of cylinder.
Whole surface of right cone	$=\pi r^2+2\pi r\times\frac{1}{2}$ slant height, where $r=$radius of base.
Surface of sphere .	$=4\pi r^2$, where $r=$radius.

71. For the measurement of irregular areas several plans are adopted:—

(1) The whole may be divided into triangles whose areas may be separately estimated.

(2) The whole may be divided into squares, or drawn on paper ruled in squares—the number of complete squares counted, and the area of incomplete ones estimated.

(3) Or we may use Simpson's rules, which will be given in the next lesson.

(4) The figure may be cut out in cardboard or thin metal, and weighed, when its area will be found by comparison with the weight of a definite area.

(5) Or we may use the planimeter, an instrument which enables us to obtain directly the area enclosed by any curved boundary. We shall presently describe such an instrument.

LESSON XXVI.—Applications of Methods of Area Measurement.

72. *Exercise.*—To find by several methods the area enclosed by a curve traced on a sheet of cardboard.

Apparatus. — Pins, string, mathematical instruments, glass millimètre scale, cardboard, balance, etc.

First Method.—Let us suppose an ellipse to be drawn in the following manner:—Insert two upright pins into a piece of cardboard at a distance apart of, say, four inches. Take a piece of thread longer than four inches, and tie its ends to the pins. Place a pencil so as to bear against the string, and let it trace a curve, guided by the tightened string.[1] Now draw the major and the minor axes, and measure the length of each. The area of the ellipse will be $\frac{\pi}{4}ab$, where a is the major and b the minor axis.

Example.—An ellipse was drawn having a major axis $= 175 \cdot 2$ mm., and a minor axis $= 144 \cdot 3$ mm. Hence its area will be $3 \cdot 1416 \times 87 \cdot 6 \times 72 \cdot 15 = 19856$ square millimètres.

Second Method.—The oblong piece of cardboard on which the ellipse was drawn was weighed and measured; the ellipse was then carefully cut out and weighed. Its area may then be determined from the following proportion: As the weight of the oblong is to its area, so is the weight of the ellipse to its area.

Example.—By this method the area of the above ellipse was found to be $= 19815$ square millimètres.

Third Method.—Let us divide the major axis into a number of equal parts, and through the points of division draw ordinates so as to be intercepted by the curve; then measure the length of these ordinates and the distance between them. The following are Simpson's rules as applicable to a curved area bounded by two parallel ordinates, the 0^{th} and n^{th}:—

Rule I. Add together the halves of the extreme ordinates and the whole of the intermediate ordinates, and multiply the result by the common interval.

Rule II. Or add together the extreme ordinates, twice every even intermediate ordinate, and four times each remaining ordinate, and multiply by $\frac{1}{3}$ of

[1] A better way is to use elliptical compasses.

the common interval, the number of intervals being even.

Rule III. Or add together the extreme ordinates, twice every third, sixth, ninth, etc., intermediate ordinate, and thrice each remaining ordinate, and multiply by $\frac{3}{8}$ of the common interval, the number of intervals here being a multiple of three.[1]

Example.—The above ellipse had its major axis divided into 16 equal parts.

Here the extreme ordinates will be zero.

LENGTH OF ORDINATES IN MILLIMETRES.

No. 0	0	No. 9	142·9
1	68·7	10	139·2
2	94·6	11	133·6
3	111·9	12	124·5
4	124·6	13	112·0
5	133·7	14	94·7
6	139·5	15	68·8
7	143·0	16	0
8	144·3		

From which it will be seen that the interval between any two ordinates is $\frac{175 \cdot 2}{16} = 10 \cdot 95$.

Calculations were then made by Simpson's rules. Rule I. gave for the area 19447, showing an error of $-1 \cdot 9$ per cent as judged by the mean of Methods 1 and 2; while Rule II. gave 19641, showing an error of $-1 \cdot 0$ per cent.

Fourth Method.—In this method the polar planimeter is used. It will be desirable to give a description of this instrument.

LESSON XXVII.—The Planimeter.

73. *Apparatus.* — The polar planimeter, invented by Professor J. Amsler of Schaffhausen, will now be described. It consists (Fig. 42) of an instrument with two arms, AB,

[1] For explanation of these rules see Williamson's *Integral Calculus*, chapter vii.

BC, movable round a joint at B. At A is a point s, which is made a fixture, whilst the tracer T held by the handle h is carried round the boundary of the area under measurement. Attached to the arm BC is a wheel revolving on an axis pivoted at a and b. This axis bears a worm-pinion F, gearing on to a worm-wheel on the axis that bears the wheel W, which latter thus receives a motion

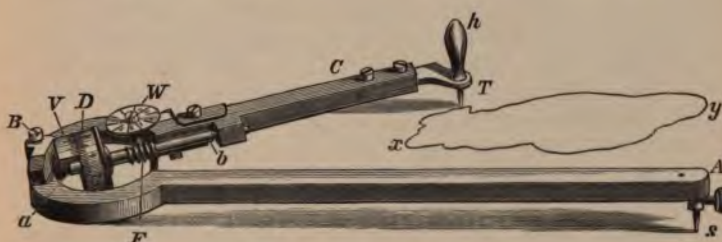


Fig. 42.—The Planimeter.

diminished ten times as the wheel D rotates. The wheel D has a raised edge, which, when the instrument is in use, rests upon the same plane that contains the area; so that, as the tracer moves, this wheel rotates (unless the motion of the tracer is wholly in the direction of the axis ab). The wheel W is graduated so as to record tens of square inches, while the large divisions of D give units, its small divisions tenths, and by means of a Vernier, V, hundredths may be estimated. To measure any area, such as xy, it is only necessary to keep the point s fixed, and bring the tracer on the curve. After reading the positions of the indices, the area is traced over and a second reading taken. The difference of the two readings will give the area directly, except in two cases to be presently considered.

Theory of the Instrument.—There are three varieties of motions of the planimeter round A, which it is well to distinguish.

(α) First suppose that BC (Fig. 43) is kept in such a position that the line from A to D is perpendicular to BC, and that in this state the planimeter is made to revolve round A. It is clear that in such a position the wheel at D will only slide and will not revolve, because the direction of motion

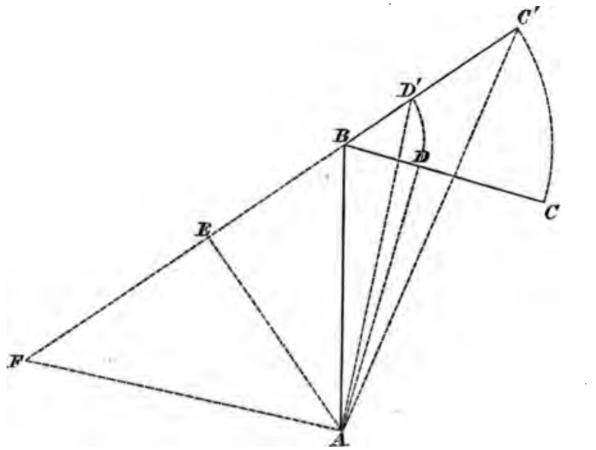


Fig. 43.

of the planimeter is perpendicular to the plane of the wheel. The point C will, of course, describe a circle.

(β) Suppose next that the arm is opened until BC is the new position of BC, and D′ that of the wheel, and that the arm is kept in this position while revolution takes place round A. The point C′ will still describe a circle; but, inasmuch as AD′ is not perpendicular to BC′, the wheel at D′ will partly move and partly slide as the angular motion of C′ goes on.

(γ) Or the instrument may start from the position BC′, but, while the angular motion of C′ round A takes place, there may likewise be a motion of BC′ round the joint at B. It is generally the last and most complicated motion of C′ which takes place when this point is made to move along the boundary of some curved or irregular figure.

In Fig. 43 produce C′B to F, and draw AE perpendicular to BE, and AF perpendicular to AD′. Now, when the instrument is moving in the method denoted by (β)—that is to say, when the point D′ is describing a circle round A, and always in a direction perpendicular to its radius AD′—let a distance = AF be in the course of time described by D′. This motion through AF may be resolved into two component motions—one through FE in a direction perpendicular to the plane of the wheel, and another through EA in a direction parallel to the plane of the wheel. It is thus clear that the wheel will slip through EF and run through EA, which latter distance will therefore be recorded on the scale of the wheel. Now, since the triangles FEA and AED′ are similar, we have the following proportion—

$$ED' : AD' :: AE : AF;$$

and hence, if AF represent in length the line described by a point with radius AD′, AE will on the same scale represent the line described by a point with radius ED′; or, in other words, while the point D′ describes say a very small arc $\delta\theta$ of a circle of radius AD′, the circumference of the wheel will move through a distance equal to the corresponding arc of a circle of radius ED′.

Now, from a well-known proposition of Euclid we have (since the angle at D is right)—

$$AC^2 = AB^2 + BC^2 - 2BC.BD,$$

also

$$AC'^2 = AB^2 + BC^2 + 2BC.BE,$$

therefore

$$AC'^2 - AC^2 = 2BC(BE + BD') = 2BC.D'E.$$

But the elementary portion of the area of the ring enclosed between the arcs traced by the points C and C′ while the small angle $\delta\theta$ is being described is equal to

$$\frac{\delta\theta}{2}(AC'^2 - AC^2), \therefore = BC\delta\theta D'E,$$

—that is to say, = a constant quantity multiplied by the distance rolled through by the circumference of the wheel.

It thus appears that the distance rolled over and registered by the wheel may be made a measure of the elementary portion of the area of the ring described by the radii AC and AC′, these being meanwhile supposed to remain constant. The circle described by the point C, when the instrument is in the position (a), is called the *datum circle*.

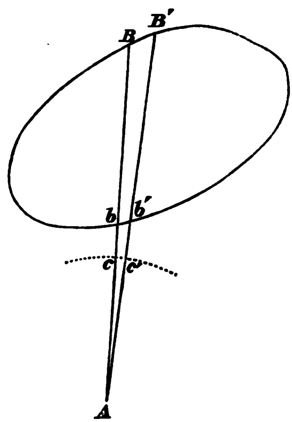


Fig. 44.

Suppose now that we wish to find the area of any figure, as, for instance, that of an oval. Let A (Fig. 44) be the centre round which the instrument moves. Let BB′ be two points very near one another in the oval. Join AB AB′, and let these lines cut the lower extremity of the oval in bb'. Furthermore, let the datum circle cc' be without the figure. As the point moves from B to B′ the wheel may turn from two causes, the first of these being the angular motion of the instrument round A, and the second the possible angular motion of the limb BC round the joint B. Let us call this latter x. Now we have seen that the former of these, when appropriate constants have been employed, may be made to

register the area $BB'c'c$, so that the wheel, as the instrument goes from B to B′, registers area $BB'c'c + x$. Again, as the point in the course of its journey moves below from b' to b, the wheel may in like manner be said to register an area equal to $- bb'c'c + x'$. Here x, x' are unknown motions of the wheel due to motions of BC round the joint at B; hence during these two passages, the upper and the lower, the area registered is

$$BB'c'c - bb'c'c + x + x' = BB'b'b + x + x'.$$

Since the same applies to other parts of the figure we have, summing up—

$$\text{Whole indication of the wheel} = \text{area of figure} + \Sigma(x),$$

the expression $\Sigma(x)$ denoting the sum of the turnings of the wheel due to the changes of angle between AB and BC, which are constantly occurring during the progress of the measurement.

But when the point has gone once round the figure, the angle at B is the same at the end as at the beginning, or, in other words, $\Sigma(x) = 0$, and hence the indications on the scale of the wheel really give us the area of the figure without any additional term.

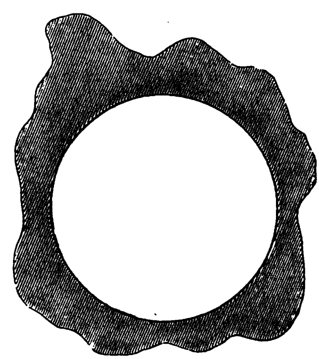


Fig. 45.

If it should happen that the datum circle lies wholly within the area, as in the adjacent figure (Fig. 45), it will, of course, be the shaded portion which is registered by the wheel; and to find the total area we shall have to add to the registered amount the area of the datum circle.

Or if it should happen that the required area lies wholly

within the datum circle, as in Fig. 46, we should have to subtract the shaded area as recorded by the wheel from that of the datum circle in order to obtain the required area. The area of the datum circle is recorded on the instrument by the maker. To verify its value it is only necessary to find the difference between the readings given by any convenient area when the fixed point and datum circle are both inside, and when they are both outside the area.

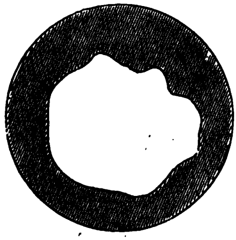


Fig. 46.

(For further information, see *Drawing and Measuring Instruments*, by G. F. Heather, in Weale's Series; Williamson's *Integral Calculus;* and Amsler's *Planimeter*, by Sir F. J. Bramwell, in British Association Report, 1872.)

Example.—The area of the ellipse already mentioned was measured several times by the planimeter, the results agreeing within ·01 square inch or 6·4 square millimètres; the mean measurement was 19807 square millimètres.

Mean Result of best Methods.—Thus we find—

Area of ellipse by first method	=	19856	square millimètres.
,, ,, second ,,	=	19815	,, ,,
,, ,, fourth ,,	=	19807	,, ,,
Mean result . . .		19826	,, ,,

VOLUME.

74. The unit of volume in the metrical system is the LITRE, which is the volume of a cubic décimètre. This unit is connected with the kilogramme, the litre being of such a capacity that it will contain a kilogramme of water at 4° C. The relation between the litre and the English system is seen in the following table:—

Table G.

Relation between English and Metrical Measures of Volume.

	English cubic inches.	English cubic feet.
Milli-litre or cubic centimètre	·061027	·0000353
Litre or cubic décimètre .	61·027052	·0353166

1 cubic inch = 16·3861759 cubic centimètres.
1 cubic foot = 28·3153119 litres.
A litre is about $1\frac{3}{4}$ pints.

75. The volume of regularly-shaped *solid bodies*, as obtained from their linear measurements, is given in the following table:—

Table H.

Measurement of some Regularly-shaped Solids.

Cylinder . .	$=\pi r^2 l$, where r = radius, l = length.
Sphere . .	$=\frac{4}{3}\pi r^3$, where r = radius.
Cone or pyramid	= Area of base × $\frac{1}{3}$ vertical height.

76. Where direct measurements cannot be resorted to, the volume of solids may be obtained by the process of first weighing the body and then dividing its weight by its density, or by noticing the volume of liquid which the body displaces.

77. The volume of *liquids* is conveniently obtained by means of graduated vessels. The chief kinds of these in use are—

1. *Measuring Flasks.*—These are made so as to contain, when filled up to a certain mark and at a definite temperature, a certain volume, which is generally

marked upon them. Litre, half-litre, and quarter-litre flasks are convenient for physical work. Fig. 47 shows a half-litre flask with the index-mark at *e*.

Fig. 47.
MEASURING FLASK.

Fig. 48.
PIPETTE.

2. *Pipettes*, or tubes open at both ends—the one opening being small, the other of such a size as to be covered by the finger. Pipettes are made of various sizes, such as those which will hold 100, 50, 25, 10, 5, and 1 centimètres. Those of small size are made of straight tubing, while the larger ones have a cylinder or bulb attached to them. When a pipette is filled by suction up to the mark *a* (Fig. 48) engraved upon it, the larger end may be covered with the finger, and the liquid conveyed and delivered without loss.

3. *Burettes.*—That known as Mohr's (Fig. 49) consists of a graduated tube provided with a glass stop-cock, and is thus a convenient instrument for delivering any quantity of a fluid.

Fig. 49.
BURETTE.

In using any of these vessels with a liquid such as water, the height of the liquid is understood to be that of the lowest point of the meniscus. To assist in reading off on the scale of the instrument the amount of liquid delivered by a burette, it is useful to employ an *Erdmann's float*. This

consists (Fig. 50) of a small piece of glass tube closed at both ends, and weighted with mercury so as to enable it to float upright. A mark engraved upon it is taken as the height of the liquid, and the position of this mark with reference to the scale of the instrument is read. The difference between the height of the float-mark before and after delivery thus gives us the amount of liquid drawn off. The use of such a float eliminates all uncertainty as to the true reading of the burette. Where a float is not used, a strip of black paper placed at the back of the burette will assist us in reading off.

Fig. 50. ERDMANN'S FLOAT.

78. *Standard Temperature of Measuring Vessels.* — The temperature at which measuring vessels are correct should be marked upon them. Suppose that a vessel is correct at 4° C.: if it be filled with water at any higher temperature the vessel will be larger than it should be, in consequence of its expansion; but, on the other hand, the water will have a density less than unity.

At a certain temperature the effect of the expansion of the vessel will be counteracted by the diminution of density of the water, and the vessel will then contain the same weight of water as it did at 4° C. For glass vessels this compensating temperature is about 10° C.

Ordinary measuring vessels in the metrical system are made so as to be correct at 15° C., which is near the average temperature of the air. The method of verifying them will be given in our next Lesson.

LESSON XXVIII.—Verification of Measuring Flasks.

79. *Exercise.*—To test the accuracy of a half-litre flask, which is marked correct at 15° C.

Apparatus.—Distilled water, thermometer, balance.

Method.—Weigh the flask dry and empty, then fill it with water up to the mark; dry the inside of the neck and weigh again. Take the temperature of the water, which should be about 15° C. If this weighing had been made at 4° C., and *in vacuo*, each gramme of water would have measured 1 cubic centimètre; but as the temperature is higher than this, and as the weighing has been made in air, we must correct for (1) the diminished density of the water, (2) the loss of weight in air.

Let W denote the apparent weight of water in air, and let σ denote the specific gravity of the air at the time of weighing; also, let 8·4 be the density of the weights. The volume of water will be nearly W cubic centimètres, and it will lose in air a weight $W\sigma$, while the weights will lose $\frac{W\sigma}{8{\cdot}4}$. Thus the total loss will be

$$W\sigma - \frac{\sigma}{8{\cdot}4} = 0{\cdot}881 W\sigma,$$

and the true weight of water in grammes will be

$$W\{1 + 0{\cdot}881\sigma\}.$$

This true weight will, however, occupy more than

$$W\{1 + 0{\cdot}881\sigma\}$$

cubic centimètres, for 1 cubic centimètre of water at 4° C. becomes at 15° 1·0009 cc. (Table N). Thus the actual number of cubic centimètres contained in the flask will be

$$W\{1 + 0{\cdot}881\sigma\}\, 1{\cdot}0009.$$

Example.—

Weight of flask and water at 15° C. =	634·5882 grms.
„ flask empty . .	135·1080 „
Apparent weight of water =	499·4802 „

Density of air at the time of weighing = ·00123.
Correction for buoyancy = 499·4802 × ·00123 × ·881 = 0·54125 grms.
Hence true weight *in vacuo* = 499·4802 + ·54125 = 500·0214 grms.

The flask therefore contains $500{\cdot}0214 \times 1{\cdot}0009 = 500{\cdot}47$ cc., or is about 0·5 cc. too large in volume.

80. Graduated tubes such as burettes, eudiometers, etc., would, if perfectly uniform throughout, contain equal volumes in equal lengths of the tube; but as this is rarely the case, it becomes necessary to "calibrate" the tube—that is to say, to find the volume corresponding to successive lengths of the tube.

This operation may best be performed by the use of mercury. Professor Bunsen's method of calibration by mercury will form the subject of our next Lesson.

LESSON XXIX.—Calibration by Mercury.

81. *Exercise.*—To calibrate a eudiometer for the measurement of gas.

Apparatus.—A reading telescope; a measuring tube *a* provided with a wooden handle (Fig. 51), one end of the tube being closed and the open end having ground edges; a ground-glass plate *c* having a ring of india-rubber cemented at its back, so that it may be held by the thumb; a thistle-headed funnel; a long wooden rod; plumb-line; mercury tray; clean mercury.

The mercury should be contained in a burette, or other vessel *b* with a stop-cock.

Method.—Fix the eudiometer, which is supposed to have been previously graduated, upright in a firm stand, with the plumb-line hanging near. Fill the measuring tube with mercury; to avoid air-bubbles the nozzle of the mercury receptacle should be at the bottom of the tube while this is being filled. When the measuring tube is quite full, close it by the ground-glass plate, and squeeze out the superfluous mercury. Transfer the measure of mercury to the eudiometer, pouring it down the funnel, which should have a long glass tube attached to it by india-rubber

Thus if we start upwards from *b*, and regard the tube between this point and the bottom as too irregular, the true value of the various graduations can easily be obtained.

Suppose we begin with graduation 23, its value may be taken as $\frac{20{\cdot}7 \times 23}{23{\cdot}3} = 20{\cdot}4$. From 23 to 44 the value of the capacity will increase by unity for each division on the tube, so that 44 will have the value 41·40.

From 44 to 64 the value of each division is 1·0147; hence the value of 45 will be 42·41, while of 46 it will be 43·43, and so on, confining ourselves to two decimal places. Division 64 will thus have the value—

$$41{\cdot}40 + (20 \times 1{\cdot}0147) = 61{\cdot}69, \text{ and so on.}$$

Here the law is sufficiently obvious.

Such a eudiometer is supposed to be read by means of a telescope, and to have mercury as its measuring fluid, in which case a table framed in accordance with such measurements as we have discussed will give us a series of relative capacities; and for most purposes this is all that will be necessary.

If we wish, however, to convert these into absolute values, the capacity of our measuring instrument must be found. To determine this we must ascertain what weight of mercury it holds when full. The vessel should be filled, and its contents weighed several times, and the mean weight taken. Calling W the weight of the mercury, and t its temperature, the volume, V, of the measuring tube in cubic centimètres will be

$$V = \frac{W \times (1 + {\cdot}0001815t)}{13{\cdot}596},$$

where ·0001815 is the coefficient of cubical expansion of mercury, and 13·596 its relative density at 0° C. This volume will be equal to 20·7 divisions of our arbitrary scale, so that one division of this scale will be $= \frac{V}{20{\cdot}7}$.

We have in the above calculation omitted the correction

due to capillarity, and before applying this it is desirable to recall to ourselves how the instrument is used in actual practice. Here Fig. 52 must be supposed to be inverted, so that *b*, instead of denoting the lowest portion, denotes the upper portion of our tube; this upper portion is then occupied by gas whose volume we wish to measure, and below it we have mercury whose convex meniscus will now be turned, with respect to the tube, in an opposite direction to that which it held during the process of calibration.

The space occupied by the gas will therefore (Fig. 53) be greater than its calibration value by twice the space between the convex meniscus *cc* and its tangent *aa*.

Fig. 53.

The value of this space can be ascertained in the following manner:—First read off the height of the meniscus in the usual way, and then pour upon the mercury a few drops of a solution of corrosive sublimate. The mercurial surface will become at once horizontal. Let its height be now again read. The difference between the two readings will give us the value of the space between *cc* and *aa*, and by doubling this we shall obtain the correction, which must be added to our calibration values in order to record the correct volume of the gas when the instrument is used in an inverted position.

CHAPTER V.

Determination of Density.

82. THE quantity of matter contained in unit of volume of any substance is defined as the **Absolute Density** of that substance. Thus if m be the mass of a body and V its volume, its density Δ will be expressed as follows:—

$$\Delta = \frac{m}{V}.$$

Relative Density is the term applied to the ratio between the masses of equal volumes of a given body and of some standard substance. Thus if m and m' be the masses of two such bodies, the latter being the standard, the relative density S of the former will be expressed as follows:—

$$S = \frac{m}{m'};$$

or, since the quantity of matter in a body is proportional to its weight, we may write $S = \frac{w}{w'}$, where w and w' are the weights of equal volumes of the substance and the standard.[1] This ratio is commonly known as the **Specific Gravity** of the substance. The term is not a good one. If we use the C. G. S. system, in which the cubic centimètre is the unit of volume, and the gramme the unit of mass, and take as our standard substance water at 4° C., the absolute becomes identical with the relative density. Thus when it is said that the absolute density of mercury at 0° C. is 13·596, it means that one cubic centimètre of that fluid will weigh at that temperature 13·596 grammes, besides meaning that the relative density of mercury at 0° is 13·596, as compared with that of water at 4° C. This is not so in the British system—the number

[1] Standard weights are in reality standard masses, and we shall treat them as such.

expressing absolute density being different from that expressing relative density. Thus if we select the grain and the cubic inch as our units, one cubic inch of water will be found to weigh 252·769 grains, which number will therefore express the absolute density of water, its relative density being, however, unity.

In consequence of this it is not so easy in the British as it is in the metrical system to derive the volume occupied by any substance from a knowledge of its weight and of its relative density.

The following Table K, which is compiled from Rankine's *Rules and Tables*, Clarke's *Constants of Nature*, and other sources, gives the relative density of some of the most important substances. As a rule the lowest and highest observed values are given.

Table K.

Relative Densities.

Solids.

Aluminium	2·50 to 2·67
Antimony	6·61 to 6·72
Bismuth	9·60 to 9·88
Copper (sheet)	8·85 to 8·89
,, (wire)	8·30 to 8·89
Carbon (graphite)	2·10 to 2·58
,, (gas carbon)	1·88
Gold	19·2 to 19·4
Iron (wrought)	7·60 to 7·79
,, (cast)	7·50
,, (wire)	7·60 to 7·73
,, (steel)	7·80 to 7·90
Lead	11·07 to 11·40
Platinum (hammered)	21·16 to 21·31
,, (wire)	21·16 to 21·53
Sodium	0·974
Silver	10·428 to 10·528
Sulphur (roll)	1·977 to 2·000
,, (flowers)	1·913
Tin	7·23 to 7·37
Zinc	6·86 to 7·20
Brass (cast)	7·80 to 8·40
,, (wire)	8·54

Solids—(Continued).

Glass (crown)	2·50 to 2·70
,, (flint)	3·00 to 3·50
Ice	0·918 to 0·920
Quartz	2·63
Sand	1·42
Sodium Chloride	2·01 to 2·20
Wood (pine)	0·50

Liquids.

Alcohol (ethylic)	0·81571	at 10° C.	
Benzine	0·883	at 15°	Boiling point 80°·4
Ether	0·7204	at 16°	
Chloroform	1·491	at 17°	,, 61°
Glycerine	1·2636	at 15°	
Carbon bisulphide	1·2931	at 0°	,, 47°·9
Sulphuric acid	1·854	at 0°	
Nitric acid	1·552	at 15°	
Hydrochloric acid	1·270		
Oil of turpentine	0·855 to 0·864	at 20°	
,, linseed	0·940		
,, olive	0·915		
Mercury	13·596	at 0°	

83. *Density of Atmospheric Air.*—Since in accurate determinations of specific gravity the weighings obtained are supposed to be reduced to what would have been their values if made *in vacuo*, a correction must be applied to them due to the density of the air in which these weighings have been made. To do this we must know the weight of one cubic centimètre of air. This may be found from the formula—

$$\text{Weight of 1 cc. of air} = \frac{\cdot 0012927}{1 + \cdot 003670t} \cdot \frac{H}{760},$$

where ·0012927 is the weight of 1 cc. of dry air at 0° C., and 760 mm. of reduced mercurial pressure at latitude 45° at the level of the sea, and where H and t are the reduced pressure and the temperature (Centigrade) of the air at the time of observation.

Finally, ·003670 is the coefficient of expansion of dry air under constant pressure.

If great accuracy is desired, the above formula must be slightly corrected for the latitude of the place of observa-

tion, and for its height above the level of the sea, these elements producing a change upon the coefficient of gravity. To find this let λ denote the latitude of the place of observation, and h the height in mètres above the level of the sea, then the weight of 1 cc. of dry air at 0° C. and 760 mm. of pressure will be equal to

$$\cdot 0012927\left\{1 - \cdot 002837 \cos 2\lambda\right\}\frac{(6366198)^2}{(6366198 + h)^2}.$$

At London (latitude 51° 28′) and at the level of the sea we shall have weight of 1 cc. = ·0012935. We shall adopt this value in the following table. If the air be not dry we must subtract from the height of the barometer $\frac{3p}{8}$, where p is the pressure of the vapour of water present in the air. The value of p is best determined by ascertaining the dew point by means of wet and dry bulb thermometers (see Vol. III.). We now give a table which will record the exact density of *dry* atmospheric air compared with water at 4° C. for different temperatures and reduced pressures recorded at London:—

TABLE L.

DENSITY OF AIR.

t.	H=720 mm.	730 mm.	740 mm.	750 mm.	760 mm.	770 mm.
10	·001182	·001199	·001215	·001232	·001248	·001264
11	·001178	·001194	·001210	·001227	·001243	·001259
12	·001174	·001190	·001206	·001223	·001239	·001255
13	·001170	·001186	·001202	·001219	·001235	·001251
14	·001165	·001181	·001198	·001214	·001230	·001246
15	·001161	·001178	·001194	·001210	·001226	·001242
16	·001158	·001174	·001190	·001205	·001222	·001238
17	·001153	·001169	·001185	·001201	·001217	·001233
18	·001149	·001165	·001181	·001197	·001213	·001229
19	·001145	·001161	·001177	·001193	·001209	·001225
20	·001142	·001157	·001173	·001189	·001205	·001221

PROPORTIONAL PARTS.

1	2	3	4	5	6	7	8	9 mm.
2	3	5	6	8	10	11	13	14

Example.—Let $t = 18°$ C. and H = 763 mm. (as observed by the barometer reduced) for the whole body of air; further, let the vapour pressure be 6·5 mm. Here $6·5 \times \frac{3}{8} = 2·4$ has to be subtracted, and hence the reduced pressure is 760·6, while the final density is ·001214.

84. *Relation between Density and Temperature.*—Since the volume of a body varies with its temperature, its density will be different at different temperatures, and hence in stating the density of a body its temperature should be given. It is, however, convenient to reduce all density determinations to 0° C., and to make the comparison with water at 4° C. In order to apply the necessary correction, a knowledge of the coefficient of expansion of the body and of water is required. The following tables give the expansion of mercury, water, and glass:—

TABLE M.

THE ABSOLUTE DILATATION OF MERCURY AND ITS DENSITY.

True Temperature Cent. as determined by an Air Thermometer (t).	Whole Dilatation from 0° to t° C. of a Volume of Mercury equal to Unity at 0°.	Density of Mercury compared with Water at 4° C.
0°	...	13·5953
10	·001792	13·5710
20	·003590	13·5467
30	·005393	13·5224
40	·007201	13·4981
50	·009013	13·4739
60	·010831	13·4496
70	·012655	13·4254
80	·014482	13·4012
90	·016315	13·3771
100	·018153	13·3529

Table N.

The Density and Volume of Water at various Temperatures, according to M. Despretz.

Temperature Cent.	Volume.	Density.	Temperature Cent.	Volume.	Density.
− 9°	1·0016311	0·998371	15°	1·0008751	0·999125
− 8	1·0013734	0·998628	16	1·0010215	0·998979
− 7	1·0011354	0·998865	17	1·0012067	0·998794
− 6	1·0009184	0·999082	18	1·00139	0·998612
− 5	1·0006987	0·999302	19	1·00158	0·998422
− 4	1·0005619	0·999437	20	1·00179	0·998213
− 3	1·0004222	0·999577	21	1·00200	0·998004
− 2	1·0003077	0·999692	22	1·00222	0·997784
− 1	1·0002138	0·999786	23	1·00244	0·997566
0	1·0001269	0·999873	24	1·00271	0·997297
+ 1	1·0000730	0·999927	25	1·00293	0·997078
2	1·0000331	0·999966	26	1·00321	0·996800
3	1·0000083	0·999999	27	1·00345	0·996562
4	1·0000000	1·000000	28	1·00374	0·996274
5	1·0000082	0·999999	29	1·00403	0·995986
6	1·0000309	0·999969	30	1·00433	0·995688
7	1·0000708	0·999929	40	1·00773	0·992329
8	1·0001216	0·999878	50	1·01205	0·988093
9	1·0001879	0·999812	60	1·01698	0·983303
10	1·0002684	0·999731	70	1·02255	0·977947
11	1·0003598	0·999640	80	1·02885	0·971959
12	1·0004724	0·999527	90	1·03566	0·965567
13	1·0005862	0·999414	100	1·04315	0·958634
14	1·0007146	0·999285			

85. *Use of Glass Vessels in Determinations of Density.*—The expansion of the glass of the vessels used in density experiments requires to be taken into account. The following table gives the coefficient of expansion for different kinds of glass:—

TABLE O.

CUBICAL EXPANSION OF GLASS FOR 1° C. (BETWEEN 0° AND 100°).

French flint glass . .	·00002616	Lavoisier and Laplace.
,, ,, . .	·00002583	Dulong and Petit.
,, ,, . .	·00002580	Despretz.
English ,, ,, . .	·00002436	Lavoisier and Laplace.
German ,, ,, . .	·00002556	Magnus.
French tube ,, . .	·00002649	Regnault.
,, globe ,, . .	·00002592	,,
Soft soda ,, . .	·000026	Kopp.
Hard potash ,, . .	·000021	,,

As a general mean, derived from a large number of observations, we shall take the value ·000025.

RELATIVE DENSITY OF SOLIDS AND LIQUIDS.

LESSON XXX.—Determination of Relative Density by Weighing in Water.

86. *Exercise.*—To find the relative density of a brass weight.

Apparatus.—Balance, weights, silk thread, beaker, distilled water, thermometer, and either (α) a specific gravity pan—*i.e.* a balance pan with short suspending arms and hook beneath, to replace one of the ordinary pans of the balance—or (β) a wooden stool, which may be placed over the balance pan and made to rest on the balance case, so as not to interfere with the movement of the balance.[1]

Method.—Weigh the brass in the pan to be used with water, then by a thread of spun silk (see Appendix) sus-

[1] The use of a specific gravity pan will be found convenient where a large number of determinations have to be made; but where the balance is used likewise for other purposes it is better to employ the stool.

pend the brass upon the hook of the balance (see Fig. 54) or balance pan, and again weigh. Let a beaker of distilled water be boiled and allowed to cool to the temperature of the balance case. Let the brass weight hang in the beaker so as to be covered with water, and examine well, in case there are any air-bubbles on the surface of the weight. If there are any, they must be removed by a camel-hair brush. In weighing the brass, as it hangs in the water, some difficulty will be found from the adhesion of the water to the thread at the place where it cuts the water. If the weight be small, this will so *damp* the oscillations that the balance will speedily come to rest. In this case it is better not to attempt to weigh by the method of vibrations, but to adjust the weights until the pointer comes to rest at its zero point. When the weighing is completed, take the temperature of the water.

Fig. 54.

Example.—

Weight of brass in air	= 49·9995	grammes.
,, ,, and thread	= 50·0000	,,
,, in water at 13°	= 44·0373	,,
Loss of weight in water	= 5·9627	,,

The cocoon silk has about the same density as water, so that the weight of that portion of it which is in the water will be *nil*. We shall, however, suppose that, in the present experiment, this is only a small fraction of the

whole length, so that the loss of weight will be entirely due to the brass. Hence, as a first approximation,

$$\frac{49\cdot9995}{5\cdot9627}=8\cdot38538$$

will be the relative density of the brass.

We have, however, omitted several corrections, the most important being:—

(1.) That due to the fact that at 13° C. the density of water is not unity, but is in reality (see Table N) ·99941. If we apply this correction, we find that the relative density now becomes

$$\frac{49\cdot9995}{5\cdot9627}\times\cdot99941=8\cdot38043.$$

(2.) The correction due to buoyancy of air must likewise be considered, the weighing not having been made *in vacuo.*

Since, however, the first weighing was made against standard brass weights, there will be here no correction, but, in the second weighing, namely that of the brass in water, the apparent weight, 44·0373, will require correction on account of the loss of weight of the brass weights in air, and this loss will amount to

$$\frac{44\cdot0373}{8\cdot4}\times\cdot0012,$$

where 8·4 is the approximate density of the weights and ·0012 a sufficiently exact approximation to the density of air, so that the true weight in water will be

$$44\cdot0373\left(1-\frac{\cdot0012}{8\cdot4}\right)=44\cdot0310,$$

and the true loss of weight will be 5·9685. Hence the relative density (if we assume that of the water as unity) will be

$$\frac{49\cdot9995}{5\cdot9685}=8\cdot37723\,;$$

but, if we take the density of the water to be ·99941, and thus embody both corrections, the final result will be the

True relative density of brass weight = 8·37229.

In this final result we cannot be sure of the last two places without taking further precautions, so that we may state the true specific gravity of our brass weight to be 8·372.

LESSON XXXI.—Relative Density by Weighing in Water (*Continued*).

87. *Exercise.*—To find the relative density of a platinum crucible.

Apparatus.—As in the previous lesson, but fine platinum wire to be used for suspension.

Method.—The method is the same as before, but we shall suppose that greater exactness is required, and that the temperature of the balance and the pressure of the air have been observed. In order to make sure that all air has been expelled, it is advisable to boil the platinum crucible in the distilled water in which it is to be suspended. The calculation will be arranged so as to compare platinum at 0° C. with water at 4° C.

In physical measurements much time may be saved by examining quantitatively the effect of the various corrections on the result sought, so that it may be known which are the most important and deserving of care. It will be seen from what follows how a troublesome formula may be simplified without any appreciable loss of accuracy. The investigation which will be given is due to Matthiessen.

Determination of Working Formulæ.—Let W be the apparent weight of the body in air at temperature $t°$ and pressure p mm., and W′ its apparent weight in water at temperature $t_1°$ and p_1 mm.

Let V be the volume of the body at 0° C., and let V′ be its volume at $t°$ C. If e be the coefficient of cubical expansion, we have $V' = V(1 + et)$, which may be written $V' = V\alpha$, where $\alpha = 1 + et$.

Let S be the density of the body at 0°, and S′ its density at $t°$, then $S' = \frac{S}{\alpha}$.

Let σ be the density of the air at the temperature $t°$ and pressure p.

Let ρ be the density of the water in which the body is weighed.

Let B be the density of the weights at 0°, and $\frac{B}{\beta}$ their density at $t°$, where $\beta = 1 + \eta t$, η being the coefficient of cubical expansion of brass.

Let m be the volume at 0° of the weights that were employed for the weighing in air, and m_1 the volume of those employed for the weighing in water, also at 0°, then $m\beta$, $m_1\beta$ will be their volumes at $t°$.

Two weighings have to be made:—

(1.) *Weighing in Air.*—Here the temperature and pressure of the air are, we have supposed, $t°$ and p mm.

Now, the true weight of the body *in vacuo* may be denoted by its volume multiplied by its density. Hence $V\alpha\left(\frac{S}{\alpha}\right)$ will be the true weight of the body *in vacuo*. Hence also $V\alpha\left(\frac{S}{\alpha} - \sigma\right)$ will be its true weight in air. In like manner $m\beta\left(\frac{B}{\beta} - \sigma\right)$ will be another expression for its true weight in air, so that

$$V\alpha\left(\frac{S}{\alpha} - \sigma\right) = m\beta\left(\frac{B}{\beta} - \sigma\right) \quad . \quad . \quad . \quad . \quad (1)$$

(2.) *Weighing in Water.*—Between the times of weighing in air and weighing in water the temperature and pressure of the air are supposed to have changed to $t_1°$ and p_1; suppose, therefore, that α, σ,

β, have become α_1, σ_1, β_1. Hence we find, as above,

$$V\alpha_1\left(\frac{S}{\alpha_1}-\rho\right)=m_1\beta_1\left(\frac{B}{\beta_1}-\sigma_1\right) \quad . \quad . \quad . \quad (2)$$

It follows from dividing (1) by (2) that

$$\alpha\left(\frac{S}{\alpha}-\sigma\right):\alpha_1\left(\frac{S}{\alpha_1}-\rho\right)=m\beta\left(\frac{B}{\beta}-\sigma\right):m_1\beta_1\left(\frac{B}{\beta_1}-\sigma_1\right).$$

Then since W is the apparent weight of the body in air, and W′ its apparent weight in water, $W=mB$ and $W'=m_1B$. Hence also

$$S-\sigma\alpha:S-\rho\alpha_1=W\left(1-\frac{\beta\sigma}{B}\right):W'\left(1-\frac{\beta_1\sigma_1}{B}\right).$$

Now, in this proportion, the first term will bear to the difference between the first and second the same ratio that the third bears to the difference between the third and fourth. Hence

$$S-\sigma\alpha:\rho\alpha_1-\sigma\alpha=W\left(1-\frac{\beta\sigma}{B}\right):W\left(1-\frac{\beta\sigma}{B}\right)-W'\left(1-\frac{\beta_1\sigma_1}{B}\right),$$

from which it follows that

$$S=\sigma\alpha+\frac{W\left(1-\frac{\beta\sigma}{B}\right)(\rho\alpha_1-\sigma\alpha)}{(W-W')-\frac{1}{B}\left(W\beta\sigma-W'\beta_1\sigma_1\right)} \quad . \quad . \quad . \quad (3)$$

Let us now consider how expression (3) can be simplified. In the first place,

σ is very small compared with ρ, α, α_1, β, β_1, or S;
α and α_1 are very nearly equal to unity;
β and β_1 are very nearly equal to unity;
ρ is very nearly equal to unity.

If we neglect σ, and make these other quantities equal to unity, we obtain

$$S=\frac{W}{W-W'},$$

which will give us a result differing from the truth by about 0·25 per cent.

kilogramme), pure mercury (see Appendix), distilled water, a thermometer, a stoneware trough, having in the centre a metal plate with two india-rubber bands stretched across, between which a specific gravity bottle may be held. Hot water and a wooden spoon for stirring. Also a hot-air bath, hydrochloric acid, and caustic soda, to be used for drying and cleaning the specific gravity bottle. The kind of bottle to be used is seen in Fig. 55. It is of glass, capable of holding 50 cubic centimètres, and provided with an accurately ground glass stopper. This stopper is pierced with a small hole through which the excess of liquid is spirted out when the stopper is pushed tight.

Fig. 55.

Method.—We must first make the specific gravity bottle quite clean and dry. To make it chemically clean, let it first be washed out with a solution of caustic soda to remove grease, and then with hydrochloric acid. All traces of acid must then be removed by frequent washings, first with common, and lastly with distilled water. The bottle should next be dried in a hot-air bath. The process of drying may be hastened by sucking the moist air from the heated bottle by a glass tube, and removing the drops of water by clean filter-paper. During drying, the bottle should be protected from dust, perfect cleanliness being necessary. When the bottle is dry, allow it to cool and then find the weight of it when empty.

Next fill the bottle with purified mercury at the temperature of the room, pouring the mercury down a funnel with a small opening. This must be done so that no air-specks are introduced. If any such are formed, they may be removed by causing a large air-bubble to travel along the inside of the bottle, so that it may coalesce with and remove the smaller bubbles. When the bottle has been

thus perfectly filled replace the stopper and push it tight into its place.

Now prepare the water-bath so that it may have some temperature above that of the room (20° C. is usually chosen as a convenient temperature), and place the specific gravity bottle in the centre between the india-rubber bands. Stir the liquid with the wooden spoon and keep up its temperature exactly at 20° degrees by adding as much hot water as is necessary. When the bottle has been in the bath some time, wipe off the superfluous mercury which has been driven through the capillary opening, owing to the expansion through heat of the whole body of mercury, and allow the bottle to remain in the water until no more such mercury is driven out. Carefully dry the bottle and allow it to cool. When supported by the neck, the weight of the mercury is sufficiently great to enlarge the bottle, so that the mercury no longer quite fills it. This is the best way of carrying the bottle, for otherwise it may receive pressure on its bottom or sides, which will cause overflow and consequent loss of mercury. Let the bottle be weighed when it has acquired the temperature of the balance case.

The next operation is to fill the bottle with distilled water, which has been previously boiled, the temperature being 20° as before. This is in all respects similar to that of filling it with mercury, which we have just described. When the bottle has cooled from 20° to the temperature of the air, the water will have withdrawn from the capillary orifice into the bottle itself, and there will thus be no risk of evaporation of the water while the operation of weighing is going on.

The weighings should be made with the same set of weights, by the method of Gauss or Borda, and should be as near as possible to one another with respect of time, so that the correction for temperature and pressure may be the same for both. After the heavy weighing with the mercury, the balance should be tested.

The advantages of filling the bottle with both fluids at the same temperature, this being above the temperature of the room and of the hands, are very obvious. *First*, there will be no correction required for expansion of glass. *Secondly*, accidental heating caused by handling will not affect the result. *Thirdly*, there is no risk of loss of mercury due to overflow, or of water due to evaporation.

If we suppose the air to have the same density at the times of the three weighings, it is plain that we may obtain the weight in air of the mercury or of the water by subtracting the weight of the bottle empty from that of the bottle filled. The respective weights will require to be corrected to *vacuo* in the manner already explained. What we have thus obtained will be the true weights of equal volumes of mercury and of water both at 20°. We must next find, by means of the tables of expansion already given, what will be the true weight of this same volume of mercury, the density being that which this fluid has at 0°; and also what will be the true weight of this volume of water, the density being that which this fluid has at 4°. Finally, having obtained these values, we must divide the one by the other in order to find the density of mercury.

Example.—

		grms.	
I.	Apparent weight of empty bottle	17·8513	Filled at 20° C.
II.	,, ,, bottle with mercury	696·2640	
III.	,, ,, ,, with water	67·7889	

Density of air taken as ·0012.

From this we find

Apparent weight of mercury II.-I. = 678·4127
,, ,, water III.-I. = 49·9376.

Thus the capacity of the bottle is nearly 50 cubic centimètres. Hence the loss of weight by air displacement of the mercury or water is

$$50 \times \cdot 0012 = \cdot 060 \text{ grms.}$$

Also loss of weight by air displacement of brass used in weighing mercury

$$= \frac{678}{8 \cdot 4} \times \cdot 0012 = \cdot 097.$$

Finally, loss of weight through air of brass used in weighing water

$$=\frac{49\cdot9}{8\cdot4}\times\cdot0012=\cdot007.$$

The true weight of mercury is therefore

$$=678\cdot4127+\cdot060-\cdot097=678\cdot3757\,;$$

also true weight of water

$$=49\cdot9376+\cdot060-\cdot007=49\cdot9906.$$

If therefore the comparison were made at 20°, the relative density of mercury would be

$$\frac{678\cdot3757}{49\cdot9906}=13\cdot570.$$

But as we wish to compare mercury at 0° with water at 4°, we find by Table M that unit volume of mercury at 0° becomes 1·003590 at 20°, and by Table N that a unit volume of water at 4° becomes 1·00179 at 20°. The corrected density is therefore

$$13\cdot570\times\frac{1\cdot00359}{1\cdot00179}=13\cdot594.$$

LESSON XXXIII.—Use of Specific Gravity Bottle (*Continued*).

89. Let us now consider generally the method of calculating results when the specific gravity bottle is used:—

Let W_1 be the apparent weight of empty bottle, the temperature and pressure of air being τ and π.

Let W_2 be the apparent weight of bottle and water, the temperature and pressure of air being τ' and π'.

Let W_3 be the apparent weight of bottle and liquid, the temperature and pressure of air being τ'' and π''.

Let the bottle be filled with water at t degrees of temperature, and with the other liquid at t' degrees.

Let S be the density of the liquid at t' degrees.

ρ „ „ water at t „

Let σ, σ', σ'' be the density of the air at the three weighings, as above.

B " " brass weights.[1]

g " " glass bottle.

Let k, k' be the ratio of the volume of glass at t and t', to the volume at 0°, where $k = 1 + \gamma t$ and $k' = 1 + \gamma t'$, γ being the coefficient of cubical expansion of the glass.

V " internal volume of bottle at 0°.

w " weight of bottle *in vacuo*.

The approximate volume of the glass, of which the bottle is composed, will be

$$\frac{W_1}{g}.$$

The water filling the bottle will have the approximate volume

$$\frac{W_2 - W_1}{\rho}.$$

The approximate volume of the bottle and water together, or

$$\frac{W_1}{g} + \frac{W_2 - W_1}{\rho},$$

we shall denote by V′, an approximate value only being required.

I. In the first weighing, the apparent weight of the bottle, or W_1, will require to be reduced to *vacuo*. The true weight will be

$$w = W_1 \left\{ 1 + \sigma \left(\frac{1}{g} - \frac{1}{B} \right) \right\} \quad . \quad . \quad . \quad . \quad (1)$$

as proved in Chapter III.

II. From the second weighing we have

$$w + Vk\rho - \sigma' V' = W_2 \left(1 - \frac{\sigma'}{B} \right) \quad . \quad . \quad . \quad . \quad (2)$$

w being the true weight *in vacuo* of the empty bottle, and $Vk\rho$ the true weight *in vacuo* of the water, and the members of equation (2) giving the true weights of these in air.

[1] The effect of the slight variation in the density of the weights has been neglected.

III. For the third weighing we have, in a similar manner,

$$w + Vk'S - \sigma''V' = W_3\left(1 - \frac{\sigma''}{B}\right) \quad . \quad . \quad . \quad . \quad (3)$$

IV. From (3) we have

$$Vk'S = \sigma''V' - w + W_3\left(1 - \frac{\sigma''}{B}\right) \quad . \quad . \quad . \quad . \quad (4)$$

And from (2)

$$Vk\rho = \sigma'V' - w + W_2\left(1 - \frac{\sigma'}{B}\right) \quad . \quad . \quad . \quad . \quad (5)$$

Dividing (4) by (5), we have

$$\frac{k'S}{k\rho} = \frac{\sigma''V' - w + W_3\left(1 - \frac{\sigma''}{B}\right)}{\sigma'V' - w + W_2\left(1 - \frac{\sigma'}{B}\right)} \quad . \quad . \quad . \quad . \quad (6)$$

An expression which will enable us to find S even although the bottle has been filled at different temperatures, and the weighings have been performed at times when the density of the air has been different.

If the experiment has been conducted as in the previous Lesson, the expression may be considerably simplified; for then $\sigma = \sigma' = \sigma''$ and $k = k'$ and (6) may be thrown—by inserting the values of V' and w in numerator and denominator, adding to and subtracting from the numerator

$$(W_3 - W_1)\frac{\sigma}{\rho},$$

and then collecting and rearranging the terms—first into the form

$$\frac{S}{\rho} = \frac{(W_3 - W_1)\left\{1 - \frac{\sigma}{B} + \frac{\sigma}{\rho}\right\} + (W_2 - W_3)\frac{\sigma}{\rho}}{(W_2 - W_1)\left\{1 - \frac{\sigma}{B} + \frac{\sigma}{\rho}\right\}},$$

and ultimately, without sensible error, into the following form—

$$S = \rho\frac{W_3 - W_1}{W_2 - W_1} + \sigma\frac{W_2 - W_3}{W_2 - W_1}.$$

If Σ denote the relative density at 0°, then $\Sigma = S\{1 + \epsilon t\}$ where ϵ is the coefficient of cubical expansion of the liquid.

Example.—Applying the above formula to the example of the previous Lesson, we have

$$
\begin{aligned}
W_1 &= 17{\cdot}8513 \\
W_2 &= 67{\cdot}7889 \\
W_3 &= 696{\cdot}2640 \\
\sigma &= {\cdot}0012 \\
\rho &= {\cdot}998213 \\
1+\epsilon t' &= 1{\cdot}003590.
\end{aligned}
$$

Whence

$$
\begin{aligned}
S &= 13{\cdot}5458 \\
\Sigma &= 13{\cdot}5944.
\end{aligned}
$$

Lesson XXXIV.—Relative Density of Solids in Small Pieces.

90. *Exercise.*—To find the relative density of fragments of glass by two different methods.

Apparatus.—For the first method, a specific gravity bottle with its accessory apparatus; for the second method, a light glass bucket with a bent wire, suitable for suspension (Fig. 56) by cocoon thread.

Fig. 56.

Method I.—Weigh the bottle empty, then with the fragments of glass in air. Next (the fragments remaining in the bottle) fill it with water and heat in a water-bath, in order that all air-bubbles may be expelled. When cool insert the stopper, and adjust the quantity of water in a bath at 20°, as in Lesson XXXII., then weigh.

Finally, remove the glass, fill the bottle with water at 20°, and again weigh.

General Method of Calculation.

Let W = weight of body (glass) in air.
P = weight of bottle filled with water only.
P′ = weight of bottle filled with glass and water.

Let V be the volume of the glass at $t°$ (the temperature at which the bottle is filled), and let S be its density at this temperature. Also, let σ be the density of the air and ρ the density of the water at $t°$. Finally, let B be the density of the weights.

Now, P + W will denote the combined weight of the bottle, filled with water only, and of the glass, while P′ will denote this combined weight when the glass has been pushed into the bottle so as to displace its bulk of water. Hence P + W − P′ will denote the weight of displaced water, and hence the approximate relative density will be

$$\frac{W}{P+W-P'}.$$

If the chief corrections be taken into account, we have

$$W\left(1-\frac{\sigma}{B}\right)=V(S-\sigma) \quad . \quad . \quad . \quad . \quad (1)$$

Also

$$(P+W-P')\left(1-\frac{\sigma}{B}\right)=V\rho \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2) we obtain

$$\frac{W}{P+W-P'}=\frac{S-\sigma}{\rho}.$$

From which we find

$$S=\rho\frac{W}{P+W-P'}+\sigma.$$

To reduce to 0°, S must be multiplied by $(1+\epsilon t)$, where ϵ is the coefficient of cubical expansion of the body.

Example.—Let the weight of the glass be 3·9460 grms. = W; also let P, or the weight of the bottle filled with water only, be 93·2080 grms.; and let P′, or the weight of the bottle filled with water and glass, be 95·5420 grms. Let the temperature be 20° C., at which temperature the density of water or ρ = ·99827, and let the density of the air or σ = ·0012, then we have

$$S=\cdot 99827\times\frac{3\cdot 946}{1\cdot 612}+\cdot 0012=2\cdot 4449.$$

Method II.—Weigh the fragments of glass in air; next weigh the bucket containing the fragments of glass, suspended by a silk thread in water, having first heated the beaker containing the water, in order to drive out air-bubbles. Finally, weigh the bucket alone in water.

General Method of Calculation.

Let W = apparent weight of the (body) glass in air.
w = apparent weight of the bucket in water.
w' = apparent weight of the bucket and body in water.
Hence $w' - w$ = apparent weight of the body in water.
And $W - (w' - w)$ = apparent loss of weight of the body in water.

Hence the approximate density

$$=\frac{W}{W+w-w'}.$$

If the various corrections be taken into account, using the same symbols as in last method, we have

$$(w'-w)\left(1-\frac{\sigma}{B}\right)=V(S-\rho) \quad . \quad . \quad . \quad . \quad (1)$$

also

$$W\left(1-\frac{\sigma}{B}\right)=V(S-\sigma) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$W : w'-w = S-\sigma : S-\rho;$$

and by compounding the ratios

$$W : W+w-w' = S-\sigma : \rho-\sigma;$$

which gives

$$S=\frac{W(\rho-\sigma)}{W+w-w'}+\sigma.$$

Finally, S may be reduced to 0° by multiplying by $(1+\epsilon t)$ where ϵ is the coefficient of cubical expansion of the body.

Example.—Let the weight of the glass in air be $17{\cdot}9997 = W$; let the weight of the bucket and glass in

water be $17{\cdot}3609 = w'$; and let the weight of the bucket alone in water be $6{\cdot}7074 = w$. Also, let $\rho = {\cdot}99884$ and $\sigma = {\cdot}001212$; hence

$$S = \frac{17{\cdot}9997 \times {\cdot}997628}{7{\cdot}3462} + {\cdot}001212 = 2{\cdot}4456.$$

Lesson XXXV.—Relative Density of Solids lighter than Water.

91. *Exercise.*—To find the density of beeswax by two different methods.

Apparatus.—Balance, etc. As it will be necessary to weigh the wax in water, a sinker must be provided, to which the wax may be attached. A convenient method is to use a small bottle, having as a cover a strip of brass with its two ends bent over the rim of the bottle. The brass has a hook convenient for suspension. Or we may use a disc of brass mounted on a spindle, having an eye at one end and a sharp point at the other, which point may be thrust into the wax. (Fig. 57.) Methylated spirit.

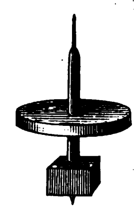


Fig. 57.

Method I.—A specimen of the wax having been selected, the operations of the last Lesson should be repeated. To expel air-bubbles it may be necessary to make use of the air-pump.[1]

Calculation.—The formula will be the same as in the last Lesson.

$$S = \frac{W(\rho - \sigma)}{W + w - w'} + \sigma.$$

[1] Air-bubbles are sometimes very difficult of removal, and in cases where the liquid and substance cannot be heated it is necessary to employ an air-pump. This is especially necessary where the body is of small bulk.

Example.—

Weight of wax in air	. . . . .	$1{\cdot}429 = W$.
,, sinker in water	. . .	$18{\cdot}141 = w$.
,, sinker and wax together in water[1]		$18{\cdot}011 = w'$.

Temperature of water, 16°·5. Density of water $= \rho = {\cdot}9988$. Density of air $= {\cdot}0012 = \sigma$.

Hence $$S = \frac{1{\cdot}429 \times {\cdot}9976}{1{\cdot}559} + {\cdot}0012 = {\cdot}9156.$$

This represents the density of beeswax at 16°·5 as compared with water at 4°, and unless the coefficient of expansion is accurately known it is better to leave the result in this form. We may write the value thus—

$$S^{16\cdot5}/_4 = {\cdot}9156.$$

This notation will be adopted in similar cases.

Method II.—Methylated spirit is lighter than beeswax, so that we may, by the successive addition of small quantities of water, raise its density to that of such wax. This suggests a very good method of finding the density of the wax. Take a piece of wax of a convenient size, place it in a test-tube containing methylated spirit, and add water until, after shaking, the wax just floats or behaves indifferently. The wax now is of the same density as the liquid. It therefore remains to determine this density, which may be done by one of the methods of the next Lesson.

Lesson XXXVI.—Specific Gravity of Volatile and other Liquids.

92. *Exercise.*—To find the specific gravity of alcohol by several methods.

Apparatus.—A specific gravity bottle of the form[2] specially adapted for volatile liquids, as shown in Fig. 58; balance, etc.; a bulb of glass containing mercury and

[1] The substance being lighter than water, w' is less than w.

[2] This simple form, provided with a cork instead of a ground-glass stopper, may be readily constructed by means of the blow-pipe. (See Appendix.)

ending in a hook (Fig. 59); Mohr's specific gravity balance; and Hare's apparatus, which will be presently described.

Method I.—Here everything is similar to that described in Lesson XXXII., except that the specific gravity bottle is provided with a stopper to prevent the evaporation of the liquid. The amount of the liquid is adjusted in the bottle until the meniscus coincides with a mark m on the neck of the bottle. When the liquid is in excess a fragment of blotting-paper is used to absorb it; when deficient, a capillary tube is made use of to introduce more liquid.

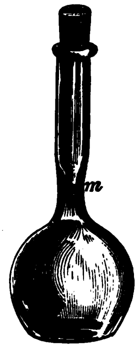


Fig. 58.

Fig. 59.

Method II.—Here a glass sinker is weighed—first in air, secondly in pure water, and finally in the liquid. The following calculations are applicable to this method:—

Let W = weight of sinker in air.
W′ = ,, ,, water at $t°$.
W″ = ,, ,, liquid.
S = specific gravity of the liquid.
δ = ,, ,, sinker.
V = volume of sinker.

Also let σ denote the density of the air, B the density of the weights, and ρ the density of the water, as before.

Then we have the following equations:—

$$\text{In air} \quad W\left(1 - \frac{\sigma}{B}\right) = V(\delta - \sigma) \quad . \quad . \quad . \quad (1)$$

$$\text{In water} \quad W'\left(1 - \frac{\sigma}{B}\right) = V(\delta - \rho) \quad . \quad . \quad . \quad (2)$$

$$\text{In liquid} \quad W''\left(1 - \frac{\sigma}{B}\right) = V(\delta - S) \quad . \quad . \quad . \quad (3)$$

From (1) and (2), by division and compounding ratios, we find

$$\frac{W\left(1-\frac{\sigma}{B}\right)}{(W-W')\left(1-\frac{\sigma}{B}\right)}=\frac{V(\delta-\sigma)}{V(\rho-\sigma)},$$

hence

$$\frac{W}{W-W'}=\frac{\delta-\sigma}{\rho-\sigma} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

And in like manner from (1) and (3)

$$\frac{W}{W-W''}=\frac{\delta-\sigma}{S-\sigma} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

hence, dividing (4) by (5), we have

$$\frac{W-W''}{W-W'}=\frac{S-\sigma}{\rho-\sigma};$$

and finally

$$S=(\rho-\sigma)\frac{W-W''}{W-W'}+\sigma.$$

This forms a very good method for obtaining the specific gravity of a liquid.

Example.—

Weight of glass bulb in air	. . .	25·193
,, ,, water	. .	22·197
,, ,, alcohol	. .	22·700

Temperature of water, 13°. Density of air, ·0012.

Hence $S=(\cdot 9994-\cdot 0012)\frac{2\cdot 493}{2\cdot 996}+\cdot 0012=\cdot 832.$

If we take $\cdot 00105=\epsilon$ as the cubical expansion of alcohol, we shall have

$$S^0/_4=\cdot 832(1+\epsilon t)=\cdot 832\times 1\cdot 01365=\cdot 843.$$

Method III.—Mohr's Specific Gravity Balance.—This instrument, which is most convenient for rapid work, is made in several forms. That of the mechanician Westphal is shown in Fig. 60. ABC is the beam, which is pivoted at B. The longer arm BC has nine notched positions at which riders may be placed. Hanging from the hook at C is a fine platinum wire supporting a small float, F, which really is

a thermometer with a single graduation mark, indicating a temperature of 15° C. The balance has been so adjusted that when the float hangs from the arm the balance is in equilibrium, and should come to rest with the indicating

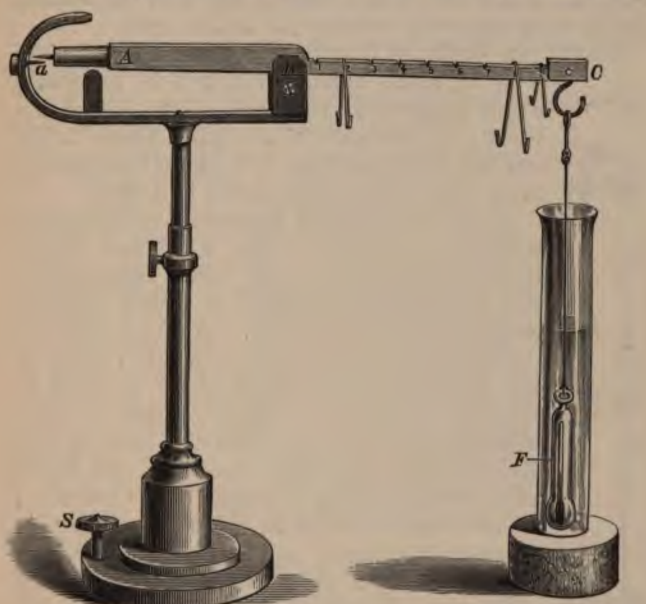


Fig. 60

points at *a* together. If this be not found to be the case on trial it is requisite to make adjustment by means of the levelling screw S. When the float is in distilled water at 15° C., equilibrium is again obtained when one of the riders (we shall call it I.) with which the instrument is provided is hung from the hook, this position being the tenth division. There are two other riders (II. and III.), one being $\frac{1}{10}$ and

adding weights until the oscillations are equal on both sides of the mark, the difficulty may be overcome. Finally, let observations be made at various temperatures, one of these being, as nearly as may be, 4° C.

Method of Calculation.—When the hydrometer floats in water up to the mark m, the weight of the liquid displaced is equal to the weight of the hydrometer.

Let V denote the volume of the instrument up to mark m, and let W be its weight; then, if weights p and p' have respectively to be added in order to adjust the instrument when immersed in two liquids of densities S and S′, we shall have

$$W+p=VS$$
$$W+p'=VS';$$

whence

$$\frac{S}{S'}=\frac{W+p}{W+p'}.$$

It is clear, however, that under the conditions of the problem, V will not be the same at any two experiments, inasmuch as the temperature being different the expansion of the glass must be taken into account. If the volume at 0° C. be called V_0, then we shall have at temperature $t°$ $V=V_0(1+kt)$, where k is the coefficient of cubical expansion of the glass of the instrument. The above equations will therefore become

$$W+p=V_0(1+kt)S$$
$$W+p'=V_0(1+kt')S';$$

hence

$$\frac{S}{S'}=\frac{W+p}{W+p'}\cdot\frac{1+kt'}{1+kt}.$$

Example.—

The instrument weighed in air . . .	65·1285
,, ,, reduced to a vacuum . .	65·2136.

The water in a large hydrometer jar was gradually cooled by the use of ice, and the additional weight required to sink the instrument up to the index-mark noted for each

temperature. The value of S was then obtained by the above formula, the value of k being taken as ·000024, and S′ being regarded as unity when $t'°$ was 4° C. The following table embodies the results:—

Temperature.	Additional weight.	Value of S.
4°	312[1]	1·000000
6·5	·300	·999756
16	250	·998766
18	·230	·998413
19	·215	·998251
21	·195	·997160
23	·165	·997301
24	·150	·997049
25	·135	·996796
26	·12	·996544
27	·11	·996367
29·5	·07	·995697
30·0	·065	·995609
31·0	·06	·995509
32·25	·05	·995326

On comparing the above values of S with those given by Despretz and other observers, it will be found that they differ from them only in the fourth place of decimals.[2] In an exact research the coefficient of expansion of the glass would have to be determined by one of the methods to be given in Vol. III.

95. The common hydrometer differs from that of Fahrenheit in being one of variable immersion, the depth to which the instrument sinks indicating the specific gravity, which is either directly recorded on a divided scale en-

[1] Calculated from observations near 4°.

[2] For a complete account of this method, and an examination into its accuracy, see *Traité de Physique Expérimentale et Mathématique*, par J. B. Biot: tome premier, 1816. This method was used by Charles for determining the density of water at different temperatures.

closed within the stem, or is expressed in arbitrary divisions whose values are found from tables.[1]

LESSON XXXVIII.—Graduation of a Densimeter.

96. *Exercise.*—To provide a hydrometer with a scale of densities. The instrument to be used for liquids heavier than water.

Apparatus.—An empty hydrometer spindle such as may be obtained from the glassworkers,[2] salt, distilled water, thermometer, apparatus for dividing a slip of paper into millimètres, mercury, sealing-wax.

Method.—Immerse the empty hydrometer in water, and add mercury until it sinks to within about an inch from the top. Drop in a fragment of sealing-wax, so that it may fall within the narrow part between the two bulbs; then apply a spirit-lamp so as to melt the wax and seal in the mercury. Next, rule a millimètre scale on a slip of stiff writing-paper, using the method described in Lesson VIII. Cut the scale so that it may be capable of insertion within the tube of the hydrometer, then insert and adjust its position until the zero line is coincident with the level of the water at 15° C. Next, prepare a solution of common salt of such a strength that the spindle, when floating in it at 15° C., will be immersed to within half an inch from the bulb. Observe the exact reading of the scale coincident with the level of the liquid, and determine precisely the density of the solution by one of the preceding methods. We shall now be able to find the value of the intermediate divisions on the hypothesis that the spindle between the two marks is of uniform diameter.

Method of Calculation.—Let V denote the volume of the hydrometer up to the zero mark, expressed in terms of the volume between two consecutive divisions taken as a

[1] A hydrometer of the former kind might well be called (following French usage) a "Densimeter."

[2] An empty Twaddell spindle will be found convenient.

unit. Let the instrument be now made to float in a liquid of density S, and let the reading be N, then the volume immersed will be $V - N$; also, we shall have $S(V - N) = W$, the weight of the hydrometer.

Again, let the instrument be immersed in a second liquid of density S′, and let the corresponding reading be N′. Then as before

$$S'(V - N') = W;$$

and hence

$$S(V - N) = S'(V - N');$$

also

$$V = \frac{SN - S'N'}{S - S'}.$$

If our first liquid be water, $S = 1$ and $N = 0$; hence

$$V = \frac{S'N'}{S' - 1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Knowing the value of V by these means, we obtain for density S″ of any third liquid the following value—

$$S''(V - N'') = V;$$

hence

$$S'' = \frac{V}{V - N''} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Example.—In distilled water the hydrometer sank to 0, and in a salt solution it sank to the 180 division of the millimètre scale. The specific gravity of the salt solution was determined at 15° C., and found to be 1·050. From these values and equation (1) we obtain

$$V = \frac{S'N'}{S' - 1} = \frac{1{\cdot}05 \times 180}{1{\cdot}05 - 1} = 3780.$$

Inserting this value in equation (2) we find

$$S'' = \frac{3780}{3780 - N''}.$$

Giving now to N″ the values 0, 5, 10, 15, etc., we obtain the following table :—

VALUE OF HYDROMETER DIVISIONS AT 15° C.

Millimètre scale. (N'').	Density.	Millimètre scale. (N'').	Density.
0	1·0000	95	1·0258
5	1·0013	100	1·0272
10	1·0026	105	1·0286
15	1·0040	110	1·0300
20	1·0053	115	1·0314
25	1·0067	120	1·0328
30	1·0080	125	1·0342
35	1·0093	130	1·0356
40	1·0107	135	1·0370
45	1·0121	140	1·0385
50	1·0134	145	1·0399
55	1·0148	150	1·0413
60	1·0161	155	1·0428
65	1·0175	160	1·0442
70	1·0189	165	1·0456
75	1·0203	170	1·0471
80	1·0216	175	1·0485
85	1·0230	180	1·0500
90	1·0244		

Having procured these values, the actual densities given by the above table should be written upon the paper scale, which may now be fixed in the proper position by a fragment of wax. It remains only then to close the open end of the instrument by the blowpipe.

Fig. 63.

97. *The Hydrometers of Baumé.*—The principles of the last lesson give us the formula for finding the value of each division in any hydrometer with degrees of equal length. We shall apply the formula to the Baumé hydrometers, which are greatly used on the Continent. There are two instruments of this name—the one being intended for liquids heavier, the other for liquids lighter than water. In the former (Fig. 63) two fixed points are obtained by immersion (1) in distilled water, which gives the zero of the scale near the top of the stem; and (2) in a solution of 15

parts by weight of common salt to 85 parts by weight of water, the point of immersion being called 15. The interval between these two fixed points is divided into 15 equal parts, and the graduation is extended downwards as far as may be desired. If the density of the water at some specified standard temperature be taken as unity, and the density of the salt solution at the same temperature be correctly known, we shall be provided with all data requisite for calculation of the value of the divisions. In the early instruments neither was the temperature of the water precisely given nor was the density of the salt solution correctly obtained. Much confusion has thus been caused, and the value of the scale divisions given by various authorities differ, as shown in the following table:—

TABLE P.

VARIATION IN VALUE OF DEGREES OF BAUMÉ'S HYDROMETER.

Baumé's Degrees.	Baumé.[1]	Gerlach.[2]	Miller.[3]	Kohlrausch.[4]
10	1·075	1·0731	1·070	1·075
20	1·161	1·1578	1·152	1·165
30	1·261	1·2569	1·245	1·268
40	1·384	1·3746	1·357	1·385
50	1·532	1·5167	1·490	1·551
60	1·715	1·6914	1·652	1·774

If we follow Gerlach and take the density of the salt solution at a standard temperature of 14° R. = 17°·5 C. as 1·11383, we obtain by the formula of the last lesson—

$$S'' = \frac{146{\cdot}78}{146{\cdot}78 - N''}.$$

Using this formula, Table Q has been calculated.

[1] From original table.
[2] Dingler's *Journal*.
[3] *Chemical Physics*.
[4] *Physical Measurements*.

TABLE Q.

VALUE OF DEGREES OF BAUMÉ'S HYDROMETER FOR HEAVY LIQUIDS[1] FOR 17°·5 C.

Deg.	Density.	Deg.	Density.	Deg.	Density.	Deg.	Density.
0	1·0000	19	1·1487	38	1·3494	57	1·6349
1	1·0068	20	1·1578	39	1·3619	58	1·6533
2	1·0138	21	1·1670	40	1·3746	59	1·6721
3	1·0208	22	1·1763	41	1·3876	60	1·6914
4	1·0280	23	1·1858	42	1·4009	61	1·7111
5	1·0353	24	1·1955	43	1·4143	62	1·7313
6	1·0426	25	1·2053	44	1·4281	63	1·7520
7	1·0501	26	1·2153	45	1·4421	64	1·7731
8	1·0576	27	1·2254	46	1·4564	65	1·7948
9	1·0653	28	1·2357	47	1·4710	66	1·8171
10	1·0731	29	1·2462	48	1·4860	67	1·8398
11	1·0810	30	1·2569	49	1·5012	68	1·8632
12	1·0890	31	1·2677	50	1·5167	69	1·8871
13	1·0972	32	1·2788	51	1·5325	70	1·9117
14	1·1054	33	1·2901	52	1·5487	71	1·9370
15	1·1138	34	1·3015	53	1·5652	72	1·9629
16	1·1224	35	1·3131	54	1·5820	73	1·9895
17	1·1310	36	1·3250	55	1·5993	74	2·0167
18	1·1398	37	1·3370	56	1·6169	75	2·0449

The hydrometer, as originally proposed by Baumé, for liquids lighter than water, has the zero of the scale at the bottom of the stem, the point being fixed by a solution of 10 parts of common salt to 90 parts by weight of water, whilst the second fixed point is obtained by immersion in distilled water. This latter point is called 10, the interval is divided into 10 equal parts and the graduation is continued upwards. As with the other instrument, the true value of a division is uncertain. We shall follow the values obtained by Gerlach, using the formula

$$S'' = \frac{145 \cdot 88}{135 \cdot 88 + N''}.$$

[1] Taken from Gerlach, Dingler's *Journal*, 1870.

TABLE R.

VALUE OF DEGREES OF BAUMÉ'S HYDROMETER FOR LIGHT LIQUIDS AT 12°·5 C.

Deg.	Specific Gravity.	Deg.	Specific Gravity.	Deg.	Specific Gravity.
10	1·0000	27	·8957	44	·8111
11	·9932	28	·8902	45	·8066
12	·9865	29	·8848	46	·8022
13	·9799	30	·8795	47	·7978
14	·9733	31	·8742	48	·7935
15	·9669	32	·8690	49	·7892
16	·9605	33	·8639	50	·7849
17	·9542	34	·8588	51	·7807
18	·9480	35	·8538	52	·7766
19	·9420	36	·8488	53	·7725
20	·9359	37	·8439	54	·7684
21	·9300	38	·8391	55	·7643
22	·9241	39	·8343	56	·7604
23	·9183	40	·8295	57	·7565
24	·9125	41	·8248	58	·7526
25	·9068	42	·8202	59	·7487
26	·9012	43	·8156	60	·7449

98. *Harmonically-divided Densimeters.*—If a densimeter could be constructed in which the successive divisions expressed equal difference of density, it would possess an advantage over the instrument whose construction we have described, for it would be much easier to read, and the use of awkward numbers would be avoided. Unfortunately, the accurate graduation of such an instrument is decidedly complicated. We know that if V be the volume of the part immersed in a liquid of density S, then $VS = W$ where W is the weight of the hydrometer. In like manner, for liquids of densities S′, S″ we have $V'S' = W$ and $V''S'' = W$, whence $VS = V'S' = V''S''$, etc. Now if S, S′, S″, etc., be a series of densities increasing in arithmetical progression such as 1, $1 + b$, $1 + 2b$, etc., we shall have for the corresponding volumes V, $\frac{V}{1+b}$, $\frac{V}{1+2b}$, etc.

Thus as the densities increase in *arithmetical* progression the volumes immersed decrease in *harmonical* progression. If, then, we wished to graduate a hydrometer so that the successive divisions should indicate successive equal increments of density, it would be necessary to divide the stem according to harmonical progression. As yet no convenient instrument has been constructed by which this may be accomplished, so that the preparation of such a

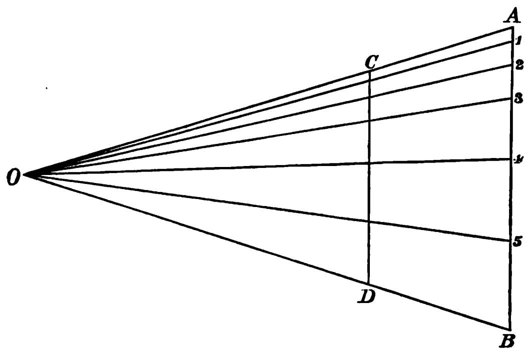


Fig. 64.

scale by the dividing engine or other means would have to be undertaken. But, when once prepared, such a scale would serve as a standard and might be adapted to any hydrometer in the following manner. The prepared scale of the standard is laid down on paper as in AB, Fig. 64 (and its ends joined to a conveniently situated point O), and a line CD (terminated by AO and BO) is drawn parallel to it of the same length as the distance between the floating points which determine the range of the hydrometer to be graduated. Lines are then drawn between O and the various divisions of the standard. Thus CD will be divided proportionately to the standard. This

operation (given AB already divided) can easily be performed by means of a simple machine, in which AO becomes a ruler with the end O pivoted, so that the movable end A may be brought in succession to the graduations of AB and the corresponding points on CD marked off.

A hydrometer much employed in England is that known as the "Twaddell," which is used for liquids heavier than water. If a Twaddell hydrometer be examined it will be seen that the maker has made the divisions closer together near the bottom in order to follow the harmonical law. The divisions do not, however, denote densities, but these may be readily deduced from them by the following simple rule—"Multiply the hydrometer reading by 5 and add 1000." Thus we have—

Fig. 64*a*.

Twaddell Degrees.	Compared with water as 1000.	Compared with water as 1.
5	1025	1·025
10	1050	1·050
15	1075	1·075
20	1100	1·100

An instrument with the complete range on one spindle, *i.e.* from 1 to 1·85, will not be sensitive. A range of this extent is much better distributed over 6 instruments with large bulbs and slender stems (see Fig. 64*a*), as in the following arrangement:—

Twaddell.	Range Twaddell Degrees.	Range Specific Gravities.
No. 1	0 - 24	1 - 1·12
2	24 - 48	1·12 - 1·24
3	48 - 74	1·24 - 1·37
4	74 - 102	1·37 - 1·51
5	102 - 138	1·51 - 1·69
6	138 - 170	1·69 - 1·85

In an instrument in which the range is small, even if the scale be not divided harmonically, but in equal parts, no great error will be caused. By this assumption the

construction of densimeters is greatly simplified, and it is by admitting it that less accurate densimeters are constructed.

99. *Technical Use of Hydrometers.*—The requirements of the inland revenue in different countries have produced a number of alcoholimeters, which are hydrometers designed to exhibit the amount of alcohol in a liquid.[1] They are empirical instruments depending upon a standard instrument graduated by immersion in liquids containing alcohol in known amount. The same is true of salinometers, saccharometers, and other hydrometers, used in technical operations. Many of these instruments are deficient in accuracy, and great confusion exists owing to the want of standard densimeters.[2]

LESSON XXXIX.—Special Methods for Density Determinations.

100. The subject of density determinations appears in so many different forms, and is capable of so many solutions, that special methods in large number have been from time to time proposed. Some of these give only an approximate value, but are useful for certain work. We proceed to give an account of the most useful of these methods.

101. *Wilson s Specific Gravity Balls.*—These are small

[1] In England the hydrometer of Sikes is employed. It differs from the hydrometers we have described in being provided with a series of weights for extending the range of the instrument. These weights when in use are placed at the lower end of the instrument.

[2] For a full account of various hydrometers see (1) the *Reports from the Secretary of the U.S. Treasury of Scientific Investigation in relation to Hydrometers*, by Professors Bache and M'Cullagh, 1848; (2) Meissner, *Die Aräometrie in ihrer Anwendung auf Chemie und Technik*, Wien, 1816; (3) *The Manual of the Hydrometer*, by Lionel Swift, R.N., London, Simpkin and Marshall. This deals especially with the salinometer.

hollow balls of glass (Fig. 65). To determine the specific gravity of a liquid by their means, it is only necessary to ascertain which one of a graduated series of such balls will just float in the liquid. It is easy to make these balls by blowing small bulbs with a short stem. They may be assorted by trial in liquids of known specific gravity, and finally adjusted by grinding. They are exceedingly delicate, and will readily show a difference in the third place in the density of a liquid.

Fig. 65.

102. *Jolly's Balance.*—Jolly proposes to use a spiral spring, suspended vertically, and having attached to its lower end two small scale pans (Fig. 66), the lower pan being attached to the upper by a fine thread or wire. Behind the pans is fixed a glass millimètre scale etched on mirror glass; the lower pan is always immersed in water. The solid whose density is sought is first placed in the upper scale pan at A, and the position of a mark, *m*, on the lower end of the spiral is then read off. The scale being on mirror glass, there will be no error due to parallax if the reflected image of the mark coincides with the mark itself, and the eye must be moved about until this position is attained. The body is then removed, and weights substituted until the mark *m* stands at the same scale division as before. The weight of the body is thus known. The body is then placed in water in B, and weights added to A, until *m* is again at the same index-mark. Dividing the weight of the body by the weight now in A, we obtain the density of the body.

Fig. 66.

By using a long spring of fine brass wire, the weighing may be performed to within ·005 milligramme, but the method is rather troublesome, owing to the oscillations of the spring.

103. *Direct Observations of Volume.*—The body, having been previously weighed, is placed in a graduated cylinder containing water, the difference of level in the water caused by immersion of the body giving us the volume.

104. *Use of Dense Liquids.* —The specific gravity of solids of not very high density, even if used in very small quantities, may be determined by diluting a liquid of high density to that point in which the solid will neither sink nor swim. A solution of the double iodide of mercury and potassium, which has a density, when saturated, of 3·11, may be used for this purpose. More recently the double iodide of mercury and barium, giving the greater density of 3·56, has been proposed for the same purpose.[1] These liquids are very suitable for separating the constituents of a mixture of minerals of different specific gravities.

Fig. 67.

105. *Liquids in Small Quantity.* —In this case it will be necessary to use small specific gravity bottles, such as may be readily made from glass tubing with the aid of the blow-pipe. Various forms are employed, the type most commonly used being shown in Fig. 67. There are several methods

[1] "Ueber eine neue Flüssigkeit von hohen specifischen Gewicht," von Carl Rohrbach, *Annalen der Physik und Chemie*, 1883. See Appendix.

of filling these small vessels—(1) the method practised in the case of thermometers; (2) the use of a funnel with a capillary stem; (3) the method shown in Fig. 68 (here

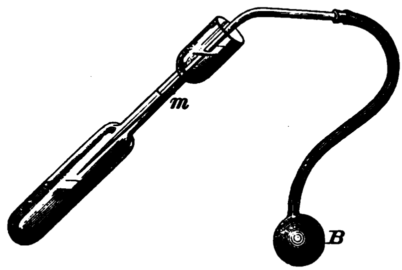


Fig. 68.

an india-rubber ball B, attached to a tube with a capillary stem, is employed to extract the air from the bottle, which is replaced by the liquid); or (4) where the vessel is provided with two openings, as in the apparatus of Fig. 69. Here it is convenient to attach one opening to an air-pump, whilst the other is immersed in the liquid.

The following method, applicable to the case of a small quantity of a volatile liquid, such as ether, was devised by Matthiessen and Hockin. A small bulb (Fig. 69) was blown with a very fine hair tube at each end, the end *a* being about ·05 mm. in diameter, the end *c* being somewhat wider. The liquid was placed in a small test-tube having its mouth plugged with cotton-wool, the test-tube itself being within a beaker (see Fig. 70). The end *a* being introduced into the liquid,

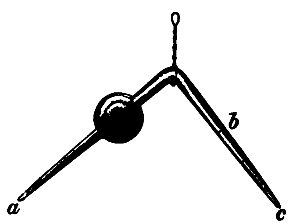


Fig. 69.

We shall now describe one of the many forms which the instrument may take. A and B (Fig. 71) are two glass reservoirs connected by a narrow portion, the upper end of A having its edges lipped and ground so that a greased ground glass plate C may fit it tightly. B is continued below into the glass tube E, which is connected by means of a T-joint with a tube D of about one mètre in length. Behind D is a graduated millimètre scale of mirror glass. At 1 and 2 are glass stopcocks.

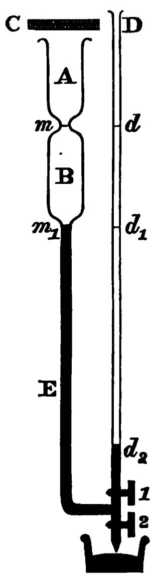


Fig. 71.
STEREOMETER.

A determination consists of several operations. (1) *Calibration.*—Mercury is poured into D until A is completely filled. Stopcock 1 is then closed, and mercury is run out by 2 until it stands at an index-mark m. The mercury that has been run off is then weighed. This will give the volume of A. In the same way the volume of B between the marks m and m_1 is found. (2) *Obtaining Levels.*—Both limbs being in free communication, the mercury is brought to m_1, and the corresponding level d_1 in D is read off. In like manner m and d are found to be at the same level. (3) *Determination of Volume.*—The mercury being at the level m d, the body whose density is required is placed in A and enclosed by C, the air within being at the pressure H, as given by the barometer. Tap 2 is then opened (1 also remaining open), and the mercury is run out until the index-mark m_1 is reached. The mercury in D will now be at say d_2, and the air enclosed will be now at a pressure less than before by d_1 d_2 millimètres. Let this difference be h, and suppose the volumes of A, B and the body to be respectively V, V′, and x. We have caused the volume $V - x$ to become $V + V' - x$, but

the volume $V - x$ was at a pressure H, and the volume $V + V' - x$ at a pressure $H - h$. Now by Boyle's law—

$$\frac{V-x}{V+V'-x}=\frac{H-h}{H}.$$

From this equation x may be obtained, and the weight of the body being known, its density can then at once be calculated.

This apparatus is liable to several sources of error, the chief being—(1) Variation of temperature during the experiment; (2) variation of atmospheric pressure; (3) presence of moisture tending to vitiate Boyle's law.

The Density of Gases.

110. The exact determination of the absolute density of a gas is a problem of great difficulty, and unsuitable for imitation in the laboratory. We shall, however, describe in outline the method used by Regnault in his classical research on this subject. For further details the student is referred to his original papers.

111. *The Absolute Density of a Gas—Regnault's Method.*—The gas, which must be perfectly pure and dry, is enclosed in a large globe of 12 litres capacity. The filling is accomplished by successive exhaustions by the air-pump, and the admission of the chemically pure and dry gas. When filled with the dry gas the globe is surrounded by melting ice until it has acquired the temperature of the ice, the gas being meanwhile at the pressure of the atmosphere. The globe is then closed, dried, and when it has attained the temperature of the air it is weighed. To avoid correction for buoyancy the weighing is performed against a globe of exactly the same size. This operation is repeated, but the gas is first reduced to a small pressure, which is recorded by a manometer. The difference between the two weights is the weight of the volume of the gas at 0° C. that would fill the globe at a pressure, which is the difference between

that recorded in the first case by the barometer and in the second case by the manometer. If the exact volume of the flask were known we should then have sufficient data for determining the absolute density. To find this volume the flask was filled with water, many precautions being taken to avoid air-bubbles. It was then kept for some time until it had attained a uniform temperature. The weight of the flask with the water being thus ascertained, the absolute weight of the water and thence the volume of the flask was calculated.

The following table gives the density of some of the more important gases:—

TABLE S.

DENSITY OF GASES AT 0° C. AND 760 MM.

	Relative Density.	Absolute Density.
Air	1·00000	·0012932
Oxygen	1·10563	·0014298
Hydrogen	·06926	·00008957
Nitrogen	·97137	·00125615
Chlorine	2·4216	·0031328
Carbon Monoxide	·9569	·0012344
Carbon Dioxide	1·52901	·0019774
Sulphur Dioxide	2·1930	·0027289
Marsh Gas	·559	·000727

112. Other methods for the measurement of the density of gases are—

(1) The Method of Dumas.
(2) The Method of Gay-Lussac.
(3) The Method of Hofmann.
(4) The Methods of Victor and Carl Meyer.

They are of greater interest to the chemist than the physicist.[1]

[1] For a description of them the student is referred to the *Treatise on Chemistry* by Professors Roscoe and Schorlemmer, vol. iii. part i.

CHAPTER VI.

Elasticity, Tenacity, and Capillarity.

113. THIS chapter will contain measurements connected with some of those properties of bodies which are determined by the relative nearness and position of their molecules, as well as other measurements intimately related to molecular constitution.

I.—ELASTICITY.

114. Elasticity is that property in virtue of which a solid body tends to recover its size and shape, and a fluid body its size, after having been submitted to the action of a disturbing force. There is for every solid body a limit beyond which, if it should be deformed, it will not entirely recover itself; while if the disturbing force act within this limit, the body will, on its withdrawal, return to its previous size and shape. This is called *the limit of perfect elasticity*, and in all engineering structures it is essential not only that this limit shall not be exceeded, but likewise that it shall not be too nearly approached.

In the theory of elasticity [1] a change in the size or shape of a body is called a *strain*, while the force in the interior of the body producing this is called a *stress*, the force *per unit of area* across any section of a body

[1] See Thomson and Tait's *Elements of Natural Philosophy*, vol. i. p. 231.

being called *the stress on this section.* Now, inasmuch as the displacements which we are here considering are small, the strains produced are proportional to the stresses producing them, and hence the relation $\frac{\text{stress}}{\text{strain}}$ forms the general expression for the coefficient of elasticity of a body, this coefficient being greatest in those cases where a very small displacement requires a very large force to produce it.

In general, twenty-one coefficients are required for a complete theory of elasticity in bodies whose structure is not uniform; but if the body under consideration be *isotropic*—that is to say, if it have the same properties in all directions—its elastic quality may be fully determined by two coefficients known as those of *elasticity of volume* and *simple rigidity.*

115. *Elasticity of Volume.*—This is measured by the amount of force per unit area applied to the body to compress it (after the manner of a fluid pressure), divided by the diminution in bulk per unit of volume which this produces. Thus if V be the original volume, which becomes $V - v$ when subjected to a uniform pressure P for unit area of surface, then the strain or compression per unit volume is $\frac{v}{V}$, and therefore

$$\text{Coefficient of elasticity of volume} = \frac{\text{stress}}{\text{strain}} = \frac{P}{\frac{v}{V}} = \frac{PV}{v} = k.$$

This is the only coefficient of elasticity possessed by liquids and gases. Its reciprocal is known as the *compressibility.*

116. *Simple Rigidity.*—But while this form of elasticity is common to matter in all its states, that of simple rigidity is confined to solids, which bodies differ from liquids inasmuch as they resist not merely a change of volume, but likewise a change of shape. In order to determine the pure coefficient of simple rigidity we must conceive some distortion of a solid that shall be unaccompanied by change of volume. Such

a distortion is denoted by a *shear* or *shearing strain*, while the internal force which produces it is called a *shearing stress*.

To realise the nature of this displacement imagine that we have a pack of cards lying on the table in the position of Fig. 72, and that by the tangential force of the hand applied to the top of the pack the cards are forced into the position denoted in Fig. 73, the depth AB remaining the same. Here it is abundantly evident that there is no change of volume, but that the space occupied by the cards in the two positions is the same. Of course it will be very easy to cause this sliding or slipping motion of the cards upon one another, for although the various particles of any one card are strongly bound together, yet the surface of one card is capable of easily sliding over that of another. Imagine, however, that the various cards are bound together by a force as strong as that which binds together the various particles of any one card, and it will then be exceedingly difficult to produce the above displacement.

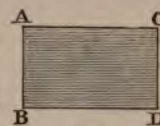


Fig. 72.

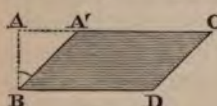


Fig. 73.

We shall now be dealing with an isotropic solid body, and the displacement we have produced will be a shear. It is evident that, in consequence of this shear (Fig. 73), the particle that, if undisturbed, would have been at A is now found at A′; it has therefore been displaced through the distance AA′, and this displacement has taken place in a solid whose depth is AB. The ratio of the displacement to the depth, or $\frac{AA'}{AB}$, may be taken as a measure of the shear.[1]

To see this imagine that AA′ is doubled while the depth remains the same, then the force resisting the strain will be doubled likewise; or if, while the distance AA′

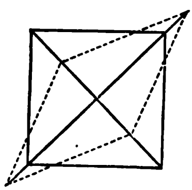
[1] The above illustration has been given with the view of making the first conception of a shear as simple as possible. In truth, however, our illustration represents a shear and a superadded motion, chiefly

remains constant, the depth be halved, the force resisting the strain will again be doubled, for it will be twice as difficult to produce the displacement in a solid of half the depth. The force called into play depends in fine on the change produced in the angle at A′, which was originally a right angle, or $\frac{\pi}{2}$. Now the angle at A′ or BA′C′ $= \frac{\pi}{2} +$ ABA′ $= \frac{\pi}{2} + \frac{\text{AA}'}{\text{AB}}$ nearly—since the change is small, and the tangent of a small angle may be used to denote the angle itself. It will thus be seen that if we measure the shearing strain by the change produced in the right angle of the solid we obtain an expression which will be proportional to the force of restitution which this strain calls into play, and therefore proportional to the stress which is in equilibrium with this force. We have thus the characteristics of a good definition, and our task is then limited to the finding of the coefficient—that is to say, of the ratio between the force of restitution (equal to stress) which the strain calls into play, and the strain itself measured as above. If we call this coefficient n we find

$$\text{Coefficient of simple rigidity} = n = \frac{\text{shearing stress}}{\text{shearing strain}}.$$

116a. *Torsion.*—We may use twist or *torsion* as a convenient means of exhibiting simple rigidity, for the element

of the nature of a twist. In a simple shear the shortened diameter lies wholly within the corresponding unshortened one, both being in the same line, while the lengthened diameter lies with its extremities without those of the corresponding unlengthened one, both being in the same line. The adjacent figure represents a simple shear, the distorted solid being indicated by dotted lines. It will be seen from this figure that, *in the first place*, the corresponding diameters of the distorted and undistorted solid are both in the same line; and, *in the next place*, that for one of these, the extremities of the distorted diameter lie symmetrically *within* those of the undistorted diameter, while for the other the extremities of the distorted lie symmetrically *without* those of the undistorted diameter.



of rotation which it introduces does not essentially alter the conditions of the problem.

Imagine therefore a great number of circular paste-board discs all of the same size to replace the pack of cards of the previous illustration, and to be built up into a coherent cylinder as in Fig. 74, the upper end being rigidly fixed. Let a vertical black line, BA, be drawn on the outside of this cylinder. Now imagine each disc to be slightly twisted round as regards that above it in the direction of the hands of a watch, and each to the same amount. The black line BA that was vertical in the undisturbed position of the system will now lie spirally in the direction BA′. Now if we imagine the various discs to be bound together by a force of the same intensity as that which binds together the various particles of any one disc, the system will represent an isotropic solid. Let now a couple, of arm L and moment FL, as shown in Fig. 74, act on this isotropic cylinder so as to twist it through a small angle. Since there is equilibrium in the strained system, the moment of the couple producing torsion will be equal to the moment of the force resisting torsion; again, the shear produced will be measured—for the *outside layer*—by the circular distance AA′, divided by the depth AB, and the force of restitution which this shear of the *outside layer* of particles calls into operation will be proportional to $\frac{\text{arc AA}'}{\text{AB}}$. A similar expression may be found for any other layer.

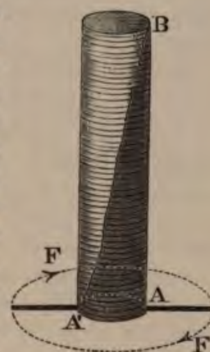


Fig. 74.

We thus see that *the stress at any point of a wire or cylinder subject to torsion is proportional to the circular displacement at that point, divided by its distance from the fixed end of the wire or cylinder.*

It now remains to find in what way the resultant stress, measured by a couple known as the "torsional couple," depends on the diameter of the cylinder or wire. In order to do this let us suppose that there are two wires or cylinders of the same substance and of equal lengths, but that the diameter of the one is double that of the other, the radius of the one being = 2, and that of the other = 1.

Let the circular cross-section of each wire be divided into a large number n of *concentric* rings of equal breadth, the breadth of each ring being in the small circle $\frac{1}{n}$ and in the larger circle $\frac{2}{n}$. Each ring in the large circle will thus, it is clear, have four times the area of the corresponding ring in the small circle. Also the circular displacement, and hence the force of restitution (when each wire is twisted through the same angle) of unit of area of a ring in the large circle, will be double that of unit area of a corresponding ring in the small circle; also the force of restitution of the first will be at a double distance from the centre of rotation as compared with that of the second; in other words, the force of the first will not only be double that of the second, but it will also act at a double leverage. Thus for an elementary ring of the large circle we shall have four times the area, and hence four times the matter, each unit of it exerting a double force at a double leverage, as compared with a corresponding elementary ring of the small circle. Hence since the same proportion holds for each elementary ring—summing up—the moment of the couple resisting torsion will, *cæteris paribus*, be 16 or 2^4 times as great in the large circle as in the small; in other words, *the moment of the torsional couple is proportional to the fourth power of the diameter of the wire.*

It remains now to show how we may in a definite manner derive the value of the coefficient of simple rigidity.

Let us therefore suppose that a torsional system of length

$= l$ and diameter $2r$ has been displaced through an angle whose arc value is equal to the radius; and let us, as before, consider one of the concentric elementary rings into which the cross-section of our cylinder or wire is supposed to be divided. Let the breadth of this ring be δx, its distance from the centre of rotation being x. Its area will be $2\pi x\delta x$, this being the difference between the ring of area $\pi(x+\delta x)^2$ and ring of area πx^2. Again, by hypothesis, the displacement at each point will be equal to radius or x. Hence the shearing strain, at a point in the ring, will be represented by $\frac{x}{l}$; and since n or the coefficient of simple rigidity $= \frac{\text{stress}}{\text{strain}}$, we shall have stress per unit area at distance x from the centre $= \frac{nx}{l}$, and for the whole elementary area—stress $= \frac{2\pi n x^2 \delta x}{l}$. But this stress acts at the leverage x, so that its moment becomes $\frac{2\pi n x^3 \delta x}{l}$. In order to find the moment of the stress for the whole cross-section we must integrate this expression between the limits $x = o$ and $x = r$, which gives us[1]

$$\text{T} = \text{moment of torsional couple} = \frac{\pi n r^4}{2l}.$$

From which it is seen that the torsional couple is directly proportional to the fourth power of the radius, and inversely proportional to the length of the cylinder, a result we have previously shown to be true from first principles.

[1] If any of our readers are not familiar with integration, the correctness of this expression may be *verified* in the following manner:—Let the radius r become $r+\delta r$, then the moment of the couple for the whole circle will become

$$\frac{\pi n}{2}\frac{(r+\delta r)^4}{l},$$

while that of the additional ring will be

$$\frac{\pi n}{2}\frac{(r+\delta r)^4}{l} - \frac{\pi n r^4}{2l} = \frac{2\pi n r^3 \delta r}{l}$$

(bearing in mind that δr being a very small quantity, powers of it above the first may be neglected). But this is the expression already obtained, only putting r instead of x. Hence the integration is correct.

The above equation gives

$$n = \frac{2l\mathrm{T}}{\pi r^4}.$$

It will be seen in Lesson XLIV. how the value of T may easily be obtained from torsion experiments, which thus serve to give us n or the coefficient of simple rigidity.

117. *Young's Modulus.*—In addition to the two coefficients already mentioned it is convenient to make use of a third, generally called "Young's Modulus," which may thus be defined. If a rod or wire of length L be stretched until it becomes of length $\mathrm{L} + l$, then $\frac{l}{\mathrm{L}}$ is the extension per unit of length. If the force causing extension be P units, and the wire or rod have a sectional area containing a units, then the force or stress is $\frac{\mathrm{P}}{a}$ per unit of cross-section, and

$$\text{Young's modulus} = \frac{\text{stress}}{\text{strain}} = \frac{\frac{\mathrm{P}}{a}}{\frac{l}{\mathrm{L}}} = \frac{\mathrm{PL}}{al} = \mathrm{M}.$$

118. *Poisson's Ratio.*—When a wire is stretched longitudinally it contracts laterally, and the ratio of proportional contraction to proportional extension has been called Poisson's Ratio. It has been contended that this ratio should be $\frac{1}{4}$ for all isotropic solids. In this case, if a unit cube were to experience an elongation e, its volume would become

$$(1+e)\left(1-\frac{e}{4}\right)\left(1-\frac{e}{4}\right) = 1 + \frac{e}{2} \text{ near}$$

Stokes has, however, remarked that this cannot hold for india-rubber, which is eminently susceptible of longitudinal extension, and offers at the same time great resistance to change of volume. For this substance we may suppose the ratio to be more nearly one-half than one-quarter, one-half expressing the ratio that would hold for a substance whose resistance to change of volume was infinitely great. While this ratio for india-rubber is thus seen to be much greater

than $\frac{1}{4}$, Thomson and Tait have shown that for cork it is much less than $\frac{1}{4}$, so that isotropic bodies present no agreement in their values for Poisson's ratio.

In order to investigate the true theory of the relation between the longitudinal extension of a stretched solid and its lateral contraction, let us begin by considering somewhat more minutely than we have already done the subject of shearing stress.[1]

Let a cube supposed to stand up from the plane of the paper suffer a shear such that A (Fig. 75) is displaced to A′ and C to C′, then, according to the principles already enunciated (Art. 116), the value of this shear will be $\frac{AA'}{AB}$. Now the shearing stress produces other effects besides this sliding, for it is evident that the diagonal A′D is less, and the diagonal BC′ greater, than its original length. Again, since we are dealing with small quantities, we may, without error, assume that the one diameter is shortened just as much as the other is lengthened, and hence what remains is to find in what proportion the diameter, say BC, is increased in length. In order to determine this let us draw CE perpendicular to BC′. Then the angle CBC′ being extremely small, EC′ will represent the actual, and $\frac{EC'}{BC}$ the increment per unit of length of BC.

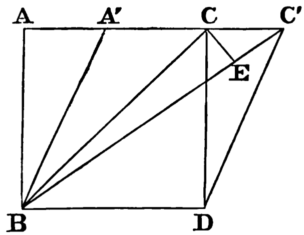


Fig. 75.

Now, since the triangle CEC′ is right-angled and isosceles, the angle CC′E being very nearly 45°, we have

$$EC' = \frac{CC'}{\sqrt{2}} = \frac{AA'}{\sqrt{2}};$$

[1] The following method of considering shearing stress is adopted from Everett's *Physical Constants*, while that of deducing Poisson's ratio is taken from Thomson and Tait's *Natural Philosophy*.

also

$$BC = AB\sqrt{2}$$

Hence

$$\frac{EC'}{BC} = \frac{AA'}{2AB}.$$

In other words, the extension or contraction of the diagonal per unit of length is equal to one-half of the shear.

It will thus be seen that a tangential force applied to an isotropic solid does two things—it produces a slipping, which is measured in the above case by $\frac{AA'}{AB}$, and a proportional alteration in length of the diagonals (which lie in a plane inclined at an angle of 45° to that of the tangential force) of $\frac{1}{2}\frac{AA'}{AB}$.

Let P denote the tangential stress per unit of area which has produced the shear, then by definition $n = \frac{P}{\text{shear}}$. Hence shear $= \frac{P}{n}$; likewise

$$\text{Change per unit of length of diagonals} = \frac{P}{2n}.$$

Suppose now the cube of Fig. 76, standing above the paper (length of side = 1), to have forces P applied to its various faces in the directions indicated, these forces being uniformly distributed so that we may represent each as acting at the middle point of its side, it is clear that the result will be a contraction of the cube in a horizontal direction, and an extension in an up-and-down direction. But from what has preceded we are entitled to expect this result to be accompanied with a tangential force acting in a plane 45° from that of AB or AC; in other words, in the plane BC for instance.

Now it is evident from the dotted lines in prolongation of the force directions that these forces may be supposed to act at the middle of BC; and since their components, perpendicular to BC, cancel one another, there only remains a tangential or slipping component equal for each to $\frac{P}{\sqrt{2}}$,

and for both to $P\sqrt{2}$. But this force is spread over the plane, whose area $= BC = AB\sqrt{2} = \sqrt{2}$, so that for unit of area the tangential force will be P.

Thus we see that forces P acting as in the figure represent

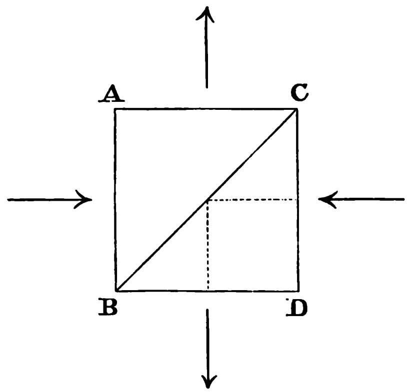


Fig. 76.

a tangential stress equal to P acting along BC, and according to what precedes this may be expected to produce a proportional extension and contraction represented by $\frac{P}{2n}$.[1]

Suppose now a cube of some given substance, lying above the paper, to suffer longitudinal extension, as in Fig. 77, by means of a force P. We may apply, without altering the conditions of equilibrium, equal and opposite forces $\frac{1}{3}$P to the four unoccupied faces of the cube, the other two being subject to the tension P. Only two of these four additions can be well represented on the plane of the paper, but those

[1] It will be noticed that in Fig. 73 the tangential force is along a side, and the extension and contraction along the diagonals, while in the above the tangential force is along the diagonal, and the extension and contraction along the sides. The reader will, however, perceive that this does not alter the physical aspect of the problem.

perpendicular to the upper and under faces, *i.e.* the faces above the paper and below it, are to be understood as acting precisely in the same way as those forces perpendicular to the side are acting in the above figure.

Now we have, in the first place, a force $\frac{1}{3}$P tending to dilate the cube in all directions; and, in the second place, two shears each of the value of $\frac{1}{3}$P, the one tending to

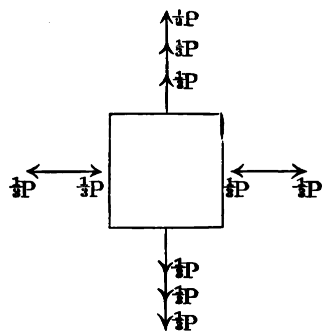


Fig. 77.

compress the cube right and left, and to extend it up and down, the other tending to compress the cube in a direction from above and below the paper, and to extend it up and down.

The force tending to dilate the cube in all directions will produce a strain amounting *per unit of volume* to

$$\frac{\frac{1}{3}\mathrm{P}}{k}=\frac{\mathrm{P}}{3k},$$

or *per unit of length* to $\frac{\mathrm{P}}{9k}$ (for the coefficient of cubical dilatation is equal to three times the linear), and each shear will produce its appropriate change of diameter

$$=\frac{\frac{1}{3}\mathrm{P}}{2n}=\frac{\mathrm{P}}{6n}.$$

The whole elongation will therefore be $\frac{\text{P}}{9k}$ for the dilating force, and $\frac{\text{P}}{6n}$ for each of the two systems of shears; in all

$$=\text{P}\left(\frac{1}{3n}+\frac{1}{9k}\right)$$

in the direction of the applied stress, while the linear contraction will be

$$\text{P}\left(\frac{1}{6n}-\frac{1}{9k}\right)$$

in all directions perpendicular to the applied stress.

Hence we find

$$\text{Lateral contraction : longitudinal dilatation} :: \frac{1}{6n}-\frac{1}{9k} : \frac{1}{3n}+\frac{1}{9k};$$

and hence

$$\text{Poisson's ratio} = \sigma = \frac{\text{lateral contraction}}{\text{longitudinal dilatation}} = \frac{\frac{1}{6n}-\frac{1}{9k}}{\frac{1}{3n}+\frac{1}{9k}} = \frac{3k-2n}{2(3k+n)}.$$

Also we see that

$$\text{M} = \frac{9nk}{3k+n}.$$

119. We give in the following table, selected from Everett's *Physical Constants*, values of the various coefficients for different substances. They are all expressed in the C. G. S. system:—

TABLE T.

COEFFICIENTS OF ELASTICITY.

Substance.	Volume Elasticity $= k$.	Simple Rigidity $= n$.	Young's Modulus $=$ M.
Distilled water	$\cdot 222 \times 10^{11}$	...	...
Glass (flint) .	3·47 to 4·15 ,,	2·35 to 2·40 $\times 10^{11}$	5·74 to 6·03 $\times 10^{11}$
Brass . . .	10·02 to 10·85 ,,	3·44 to 4·03 ,,	9·48 to 11·2 ,,
Steel . . .	18·41 ,,	8·19 ,,	20·2 to 24·5 ,,
Iron (wrought)	14·56 ,,	7·69 ,,	19·63 ,,
Iron (cast) .	9·64 ,,	5·32 ,,	13·49 ,,
Copper . . .	16·84 ,,	4·40 to 4·47 ,,	11·72 to 12·34 ,,

LESSON XL.—Young's Modulus by Stretching.

120. *Exercise.*—To find the modulus of a brass wire of about a millimètre in diameter and three mètres in length.

Apparatus.—The upper end of the wire is firmly attached by a strong brass clamp to a cross-bar in the ceiling of the room, or, better still, to a rigid support fixed into the masonry. The lower end of the wire also has a clamp with a hook, from which hangs a cage for holding weights. Under the cage is a stool (Fig. 78) with a movable top, which may be raised or lowered, with a progressive motion unaccompanied by rotation, by turning the handle. A sewing needle is fastened by wax to the wire, just above the lower clamp. A cathetometer microscope is placed on a firm slab, so that it may be focused upon the needle, whose point should be illuminated by a mirror or lamp suitably placed. A wire-gauge or other apparatus for determining the diameter of the wire will also be required, likewise a long rod with one end arranged so as to lengthen or shorten, and a set of weights.

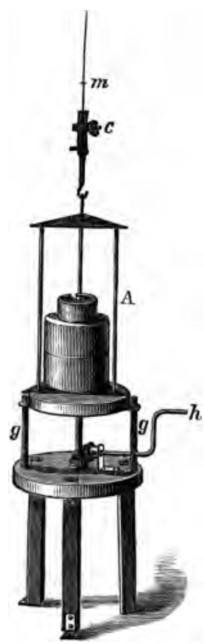


Fig. 78.

Method.—The wire is made as free from kinks as possible, and a weight, not greater than half its breaking weight, is placed in the cage. It is necessary to remark that the cage must *rest* on the stool whenever weights are added or taken away.

The breaking weight may be roughly calculated as

follows:—Obtain the diameter of the wire by the wire-gauge, and then calculate the area of the cross-section in square millimètres. This area, multiplied by the breaking weight for the substance of the wire in kilogrammes per square millimètre (see Table U), will give us the breaking weight of the wire.

When the weight referred to above has been placed in the cage, the stool is lowered, but gradually, so that the wire may not receive a sudden strain. The wire may now be left to stretch for some time. At length, when it has stretched itself sufficiently, and is as free as possible from oscillations, the cathetometer microscope is arranged so that the needle-point is seen *apparently below* the middle of the field of view. As the wire is never quite free from motion, the observer must take a reading of the exact position of the needle-point as it swings into focus; and in doing this no difficulty will be experienced if the room is tolerably free from vibration.

Now let definite weights be added. The wire will now become stretched, and the needle-point will be lowered, so that it will be seen apparently above the middle of the field of view of the microscope. Let a series of readings be taken after each successive addition and removal of weights, the whole weight on the wire being always considerably below the breaking weight. On the removal of weights the wire should return to its original position; if this be not so, either the microscope has been disturbed, or so much weight has been added that the limits of elasticity have been exceeded. This last fault must especially be avoided.

We must next find the total length of the wire. To obtain this the long pole having a sliding portion at one end is placed between the clamps, and is extended in length until the rod fits between the clamps. The point of the needle that is fixed to the wire is next used to scratch a mark on the rod; the distance between the scratch and

the end of the rod is then measured. In this way a measurement of the length of the wire may be obtained, the error of which should not exceed one millimètre.

The average diameter of the wire must now be determined. Here we have a choice of several methods :—

(1.) The diameter may be determined by an application of the wire-gauge to various parts of the wire.

(2.) Or by measurement of specimen portions by means of the compound microscope.

(3.) Or, better still, the whole length of the wire may be weighed in air and water; and having thus obtained its density, and knowing its length, its diameter may easily be determined from the formula

$$d=2\sqrt{\frac{w}{\pi\rho l}},$$

where d is the diameter, w the weight, ρ the density, and l the length of the wire.

Example.—A microscope with a 1-inch objective was used, having an eye-piece micrometer with scale divisions, each representing $\frac{1}{400}$ of an inch. Estimations were made to tenths of a scale division. The following readings were taken :—

POSITION OF NEEDLE-POINT.

Order of experiment.	Weight = 20 lbs.	Weight = 30 lbs.	Difference due to 10 lbs.
I.	11·2	32·3	21·1
II.	11·4	32·4	21·0
III.	11·4	32·35	20·95

The difference remained constant thereafter at 20·95; the elongation is therefore

$$\frac{20{\cdot}95}{400}\text{ inches}=\frac{20{\cdot}95\times 2{\cdot}540}{400}=\cdot 133\text{ centimètres.}$$

The length of the wire was 271 centimètres; the diameter was by the microscope ·0425 inches, and by the wire-gauge

·0420. Taking the radius as ·021 inches = ·05334 centimètres, the area of the cross-section is $= 3{\cdot}1416 \times ({\cdot}05334)^2$ square centimètres. We shall express the result in the C. G. S. system. Since the wire is stretched by 10 lbs. = 4535·9 grms., the stretching force in *dynes* will be 4535·9 × 981·34, where 981·34 is the value for the acceleration of gravity at Manchester.

From the formula

$$M = \frac{PL}{al}$$

we find the modulus

$$= \frac{4535{\cdot}9 \times 981{\cdot}34 \times 271}{{\cdot}133 \times ({\cdot}05334)^2 \times 3{\cdot}1416} = 10{\cdot}148 \times 10^{11}.$$

121. The method just described depends for its success on the assumption that the addition of weights does not cause any perceptible bending of the fixed support, or slipping of the wire in its clamp. If we can secure these conditions by taking care that our support is strong enough, and that an appropriate clamp is used, then the results should be accurate.

The method best adapted to eliminate this possible source of error is to have two points in the wire a measured distance apart, each having the point of a needle attached to it. The difference in vertical position of each point due to the addition of a weight is to be separately and independently read by means of two rigidly fixed cathetometer microscopes, one for each point. Thus, let A be the lowering of the upper, and B that of the lower point, then B − A will represent the elongation of that length of wire which lies between the two points.

122. *Young's Modulus by Flexure.*—Young's Modulus may also be obtained by measuring the amount of bending of a bar of known dimensions, the forces then called into action being such as afford us a mechanical means of producing longitudinal extension and compression without

altering essentially the nature of the problem. To exhibit this, let us, in the first place, suppose a rectangular bar of length L, depth d, and breadth b, to be fixed at one end and weighted at the other. The bar will become bent, as in Fig. 79. The upper portion AB′ will be extended and

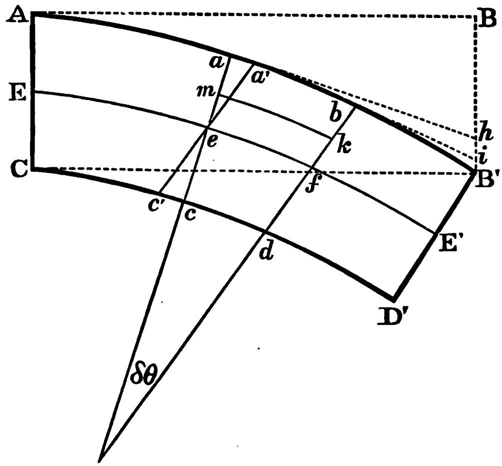


Fig. 79.

the lower portion CD′ compressed, and there will be a neutral line EE′, which we may here imagine to be half-way between the top and the bottom, the particles of which are neither extended nor compressed. Under the action of the force or weight F (applied at E′) the extremity of the bar has fallen from B to B′, through a distance which we will call l.

Let ac, bd represent transverse sections of the bar very near each other, these two lines making an angle $\delta\theta$ with each other. It is clear that if, through the neutral line at e, we draw $a'c'$ parallel to bd, we shall have aa' representing the lengthening, and cc' representing the shortening of the upper and lower fibres between the two transverse sections.

It is also clear that all the fibres above the neutral line, owing to their extension, and all the fibres below the neutral line, owing to their compression, will combine together to form a couple tending to right the bar and make it horizontal—that is to say, a left-handed couple. This will be balanced by the weight F acting as an opposing couple tending to twist the bar in a right-handed direction, its leverage being eE', which we will call x. Now, consider a fibre or layer of fibres mk, of which the vertical distance em from the neutral line is z, the depth of these fibres being δz (very small), and the breadth of the layer being that of the bar or b. Hence if we call $ef = \delta x$ (eE' being x) we have length of fibre or

$$mk = ef + \text{elongation} = \delta x + z\delta\theta.$$

Again, the cross-section of the layer is $b\delta z$. Hence if M denote Young's Modulus for the substance of the bar, the force of restitution exercised by this layer will be

$$\frac{\text{M} \times \text{cross-section} \times \text{extension}}{\text{length}} = \frac{\text{M}b\delta z \times z\delta\theta}{\delta x};$$

and the leverage of this force with reference to e being z, its moment will be

$$\text{M}\frac{b\delta\theta}{\delta x}z^2\delta z \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, in order to obtain the whole effect of the forces above the neutral line, we must integrate[1] the above expression (1) between the limits $z = o$ and $z = \frac{d}{2}$, which will give us

$$\text{M}\frac{b\delta\theta}{\delta x} \times \frac{d^3}{24}.$$

But the whole effect of the forces below the neutral line is equal in amount to the effect of those above (both tending to right the bar). Hence we must double the above ex-

[1] Our readers who are unacquainted with integration may *verify* the accuracy of the result obtained above in the way we have pointed out in the footnote of Art. 116.

pression in order to find the whole moment of the left-handed couple. But this is balanced by the force F acting at leverage x, so that finally we have the following equation of equilibrium—

$$M\frac{b\delta\theta}{dx}\times\frac{d^3}{12}=Fx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Suppose that from a we draw ah, a tangent to the curved surface of the bar at a, and that from b we draw bi, a tangent to the same surface at b, then the angle between ah and bi will be $\delta\theta$, and each of these lines (ah and bi) being very nearly equal to x, we shall have $hi=x\delta\theta$. Now the whole fall of the end of the bar $BB'=l$ may be supposed to be made up of a number of little elements of the nature of hi; in other words $hi=\delta l$, retaining our previous notation. Hence

$$x\delta\theta=\delta l \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

And if we substitute for $\delta\theta$ its value as given by equation (2), we finally obtain

$$\delta l=\frac{12F}{Mbd^3}\times x^2\delta x \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we integrate (4) between the limits $x=o$ and $x=L$, we obtain

$$l=\frac{4FL^3}{Mbd^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If we wish to use this method of flexure in order to find M, we obtain from (5)

$$M=\frac{4FL^3}{bd^3l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We see from expression (5) that the bending l for a given weight is proportional to the cube of the length, and inversely proportional to the breadth and to the cube of the depth.

123. The demonstration we have given above is for a beam secured at one end and weighted at the other; but

we may easily proceed from this to the case of a beam *supported*, not *fixed*, at the two ends, and weighted at the middle with a weight F. Here it is clear that the forces called into action will be equal in amount but opposite in direction to those that would be brought into play if the beam were hung or supported by the middle and weighted at each end with a weight $=\frac{F}{2}$. But we see that in this case we have in reality two beams fixed at one end and free at the other, each of length $\frac{1}{2}$L, and with weight $=\frac{1}{2}$F attached to the free end, so that, applying (5), the depression of either end below the centre will be

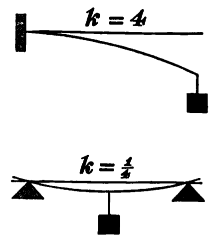


Fig. 80.

$$l'=\frac{4\frac{F}{2}\times\left(\frac{L}{2}\right)^3}{Mbd^3}=\frac{FL^3}{4Mbd^3} \quad . \quad . \quad . \quad . \quad (7)$$

Finally, in this case

$$M=\frac{FL^3}{4bd^3l'} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

On comparing formulæ (6) and (8) we see that both cases are given by

$$M=K.\frac{FL^3}{bd^3l} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $K=4$ or $\frac{1}{4}$. The two cases are illustrated by Fig. 80.

Lesson XLI.—Young's Modulus by Flexure.

124. *Exercise.*—To find the modulus for an iron bar supported at both ends.

Apparatus.—Two strong wooden supports with wedge-shaped tops, a hook for hanging weights from the bar, a sewing needle, cathetometer microscope, wire-gauge, millimètre scale, set of weights, Argand lamp or other means

for illumination. The arrangement of the apparatus is shown in Fig. 81.

Method.—Find the centre of the bar and fix by wax a needle to the edge of the bar at its centre. Place the rod on the supports with its ends equally projecting, and bring

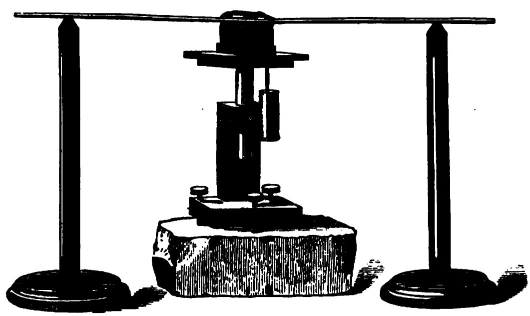


Fig. 81.

the hook which is to hold the weights so that the knife-edge in which it should terminate may be placed exactly at the marked centre. Focus the microscope on the needle-point and observe the bending produced by the addition of weights, taking observations with weights on and off successively. Obtain the dimensions of the bar.

Example.—The following readings were taken:—

Expt.	Load.	Reading.	Difference for ·5 lb.
1	carrier + ·5 lb.	31·85	18·55
	,, alone	13·30	
2	,, +1·0 lb.	25·6	18·60
	,, + ·5 ,,	7·0	
3	,, +1·5 ,,	25·4	18·60
	,, +1·0 ,,	6·8	
4	,, +2·0 ,,	27·2	18·60
	,, +1·5 ,,	8·6	
5	,, +2·5 ,,	45·90	18·65
	,, +2·0 ,,	27·25	
		Mean	18·60

Length of bar $= 79 \cdot 5$ cm. $= L$.
Depth ,, $= \cdot 46736$ cm. $= d$.
Breadth ,, $= 1 \cdot 275$,, $= b$.

A weight of $\cdot 5$ lb. $= 226 \cdot 8$ grms. is equivalent to a bending force of $226 \cdot 8 \times 981 \cdot 4$ dynes $= F$. The deflection was $18 \cdot 6$ divisions on micrometer scale, or $\cdot 1181$ cm. $= l$.

By the formula of Art. 123

$$M = \frac{1}{4} \frac{226 \cdot 8 \times (79 \cdot 5)^3 \times 981 \cdot 4}{\cdot 1181 \times 1 \cdot 275 \times (\cdot 46736)^3} = 18 \cdot 18 \times 10^{11}.$$

A second experiment made with the same bar, but with $L = 64 \cdot 34$ cm., gave $l = \cdot 0635$ cm., and $M = 17 \cdot 93 \times 10^{11}$.

Lesson XLII.—Young's Modulus by Flexure—(*Continued*).

125. *Exercise.*—To find the modulus for an iron bar by the bending of the bar fixed at one end only.

Apparatus.—Same as in the previous lesson, but an iron

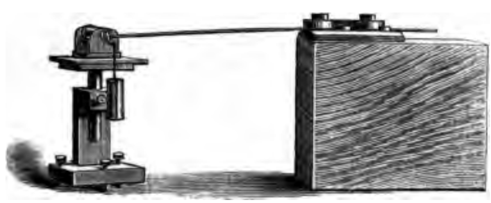


Fig. 82.

clamp, fixed to a wooden block, to be used to secure the bar firmly at one end. The arrangement shown in Fig. 82 is a good one; here the bar is held between two iron plates, the lower one being fixed to the block, whilst the upper one can be strongly screwed down so as to bind the rod placed between the two plates.

Method.—The bar being in position, a needle is fixed by wax to the edge at its extreme end. A suitable hook with a knife-edge is arranged so that the bearing point of the suspended weights may be as near the extremity of the rod as may be possible. Observations are then made as just described.

Example.—Bars of the dimensions $d = {\cdot}46736$ cm., $b = 1{\cdot}275$ cm., but of different lengths, were used. The following were the results obtained:—

Expt.	L cm.	P grms.	l cm.	M.
1	72·0	28·27	·186	$17{\cdot}11 \times 10^{11}$
2	60·6	56·61	·21617	$17{\cdot}57 \times 10^{11}$
3	52·2	111·313	·2693	$17{\cdot}73 \times 10^{11}$

126. *Hooke's Law.*—When a body is distorted within the limits of perfect elasticity, the force with which it reacts is simply proportional to the amount of distortion. As a consequence of this important law, if the constraint be at once removed, the oscillations that are set up are isochronous. We proceed to an experimental proof.

Lesson XLIII.—Isochronism of Torsional Oscillations.

127. *Exercise.*—To find the times of oscillation of a torsion system for different amplitudes.

Apparatus.—A piece of brass wire (No. 18 B. W. G.) about two mètres in length, having its upper end held by a clamp supported from a bracket (Fig. 83), while its lower end passes through a small eye in a cylindrical weight, and is there secured. A light index fixed to the weight moves

over a graduated circle. A chronometer beating half-seconds is also required.

Method.—Twist the wire through a known angle, as shown by the index, and set the system swinging. It is required to find the time of a single oscillation—that is to say, the time occupied by the system in passing between two consecutive turning points.

Since the index, when at its turning point, will have a very slow motion, it will be impossible to record the exact moment when it reaches this point. Instead of this we should record the exact moment of time at which it passes across a point about midway between two turning points, since here the motion will be most rapid. Instead of simply counting a great number of swings and noting the time occupied, we shall employ the **Method of Passages**, which will receive further explanation when the magnetometer is considered.

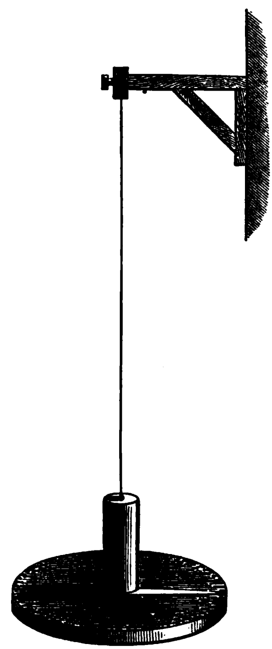


Fig. 83.

The observations should be arranged in a systematic manner; the observer is therefore advised to rule his observation book after the manner of the following form :—

DETERMINATION OF TIME OF OSCILLATION.

Passages to Right.								Passages to Left.							
	(1)			(2)		(2)-(1)			(3)			(4)		(4)-(3)	
	m.	s.		m.	s.	m.	s.		m.	s.		m.	s.	m.	s.
0	*13*	*53*	50	*18*	*41·5*	*4*	*48·5*	5	*14*	*21·5*	55	*19*	*10*	*4*	*48·5*
10	*14*	*50*	60	*19*	*39·5*	,,	*49·5*	15	*15*	*19*	65	*20*	*8*	,,	*49·0*
20	*15*	*48*	70	*20*	*37·0*	,,	*49·0*	25	*16*	*17*	75	*21*	*6*	,,	*49·0*
30	*16*	*46*	80	*21*	*35·0*	,,	*49·0*	35	*17*	*15*	85	*22*	*4*	,,	*49·0*
40	*17*	*44*	90	*22*	*32·5*	,,	*48·5*	45	*18*	*12·5*	95	*23*	*1*	,,	*48·5*
				Mean		*4*	*48·9*					Mean		*4*	*48·8*

Amplitude at commencement, *90°*
,, ,, end *10°*

Mean 50°

Mean of mean times of 50 oscillations . . . *4* *48·85*
= *288·85*
2
100 | 577·70

Time of single oscillation 5·777 seconds.

Before proceeding to the regular observations it will be useful to ascertain the approximate time of 5 oscillations by noting that of, say, 20 oscillations and dividing by 4. Thus if 20 oscillations are made in 116 seconds, the time of 5 oscillations will be 29 seconds. Then the observer should note down the exact time of a passage across a distinctly marked line or point near the middle of the range as the index moves to the right. Let this be $13^{\mathrm{m}}\ 53^{\mathrm{s}}$. We know that a fifth passage after this will take place at about $14^{\mathrm{m}}\ 22^{\mathrm{s}}$, the two passages being in opposite directions. Now a little before the moment of the expected passage observe the chronometer accurately, and continue counting the beats until the passage takes place. It will be to the left, and it should be recorded at the top of

column 3 of the table, from which it will be seen that the actual time was $14^{\mathrm{m}}\ 21^{\mathrm{s}}{\cdot}5$. The next passage to be observed will be the tenth, which will be a passage to the right. Again 29 seconds must be added to the last recorded time, and the observer must be prepared to note as before the exact moment of the passage. The processes of addition, taking time from the chronometer, noting passages, and recording, must be continued until the table is filled up.

To the unpractised observer the fact that the chronometer beats half seconds will present a little difficulty. If every tick be counted he is apt to commit mistakes. A better plan is to count alternate ticks. This can be more readily done if the counting is performed so that there are virtually always two syllables to be pronounced. One way of doing this would be as follows:—

	Also (from 1 to 20)—
	Six*ty*-one.
Twen*ty*-one.	Six*ty*-two.
Twen*ty*-two.	:
etc.	Seven*ty*-one.
	Seven*ty*-two.
	etc.

The syllable **Twen** should be pronounced with emphasis at the moment 20·5, and the syllable **one** at the moment 21·0, and so on. The main elements of success in observations of this nature are concentrated attention, coolness, and a methodical recording of results.

The observer is not, in this exercise, expected to attempt to observe a smaller interval of time than half a second; it will be afterwards explained how greater accuracy may be obtained.

With two observers the process can be performed without difficulty, one noting the passages by a series of smart taps, while the other records the exact time of these by the chronometer. Several other devices will suggest themselves with the view of increasing ease of observation. A

good plan is to attach a small mirror to the wire, so arranged that the light of a lamp is reflected into a telescope at each passage. The flash of light in the telescope affords a good signal to the observer. Or instead of using a telescope the reflexion may be made to pass over a mark on a screen.

We have supposed that observations of every fifth passage have been continued by one or other of these methods until the 95th passage. If we then subtract the 10th from the 60th, the 25th from the 75th, and so on, and write the differences in the appropriate columns, we shall be furnished with a series of values of 50 oscillations. We must then find the mean of each column, and after that the mean of these means. Finally, we must convert the result into seconds and obtain the time of a single vibration by multiplying by 2 and dividing by 100.

If the observations be repeated with the same system, but with a different amplitude, very nearly the same time will be obtained. For example, with the same wire, but with an extremely small amplitude, 5·778 seconds instead of 5·777 seconds was found to be the time of oscillation.

128. The primary intention of the last exercise was to exhibit the isochronism of torsional vibrations, which leads at once to the conclusion that the moment of the torsional couple called into action is proportional to the circular displacement. The same method might be extended to prove the other torsional laws by obtaining the time of vibration with wires of different lengths and diameters.

We have here the well-known formula

$$t=\pi\sqrt{\frac{\mathrm{I}}{\mathrm{T}}},$$

when t is the time, I the moment of inertia of the system with respect to the axis of rotation, and T the value of the

torsional couple called into operation by a unit twist. The value of T is therefore

$$T=\frac{\pi^2 I}{t^2}.$$

We can now determine the coefficient of simple rigidity, for we have seen (Art. 116) that

$$n=T\frac{2l}{\pi r^4},$$

or, substituting in this the above value of T,

$$n=\frac{2\pi Il}{t^2 r^4}.$$

From this expression the value of n will now be experimentally determined.

LESSON XLIV.—Determination of Simple Rigidity.

129. *Exercise.*—To find the simple rigidity of a brass wire.

Apparatus.—The same as in the last lesson, but with the addition of a wire-gauge or other apparatus for determining the diameter of the wire; also a long measuring rod, a millimètre scale, and a pair of callipers.

Method.—Find the diameter of the wire, its length, and the radius of the attached cylinder, the weight of which is supposed to be known. These facts, together with a knowledge of the time of vibration of the system, will give us the data necessary for obtaining the coefficient. Care must be taken that the wire be not subjected to any undue longitudinal strain.[1]

Example.—

Expt. I.—Radius of cylinder = 3·8 cm. = R.
Mass ,, = 8997 grms. = M.

[1] Sir William Thomson has shown that a wire which has been subjected to tension has its rigidity considerably altered.

The moment of inertia of the system (see Chap. VIII.) is $\frac{1}{2}MR^2$, hence

$$I=\frac{8997\times(3\cdot8)^2}{2}.$$

Also

$$\begin{aligned}\text{Time of vibration} &= \overset{s.}{5\cdot822}=t.\\ \text{Radius of wire} &= \cdot05334\text{ cm.}=r.\\ \text{Length of wire} &= 260\cdot55\text{ cm.}=l.\end{aligned}$$

Hence

$$n=\frac{8997\times(3\cdot8)^2\times260\cdot55\times2\times3\cdot1416}{(5\cdot822)^2\times(\cdot05334)^4\times2}=3\cdot881\times10^{11}$$

Expt. II.—With $l=245\cdot3$ $t=\overset{s.}{5\cdot6606}$ was found $n=3\cdot862\times10^{11}$.

130. *Elasticity of Volume.*—The coefficient of this elasticity, if M and n be previously known, can be calculated from the formula

$$k=\frac{nM}{3(3n-M)}$$

since

$$M=\frac{9nk}{3k+n}\text{ (Art. 118).}$$

Thus if the wires selected for our determinations of M and n in the preceding lessons had been precisely similar, we could have utilised the values obtained for the calculation of k. Several of the values in the first column of Table T have been calculated by the aid of this formula. The direct experimental determination of k is a problem of great difficulty, and unsuitable for imitation by the student.

LESSON XLV.—Poisson's Ratio.

131. *Exercise.*—To investigate Poisson's ratio in the case of india-rubber.

Apparatus.—About a yard of solid india-rubber about half-inch in diameter, either square or circular in section; two millimètre scales on mirror glass; two clamps, weights, and a hook to support them. One clamp is fixed to

the wall, and is used to support the india-rubber; the other clamp is provided with a hook, from which the weights are hung. The scales are fixed behind the india-rubber, one being near each clamp (see Fig. 84). A needle is thrust through the substance of the india-rubber at *a* and another at *b*. A wire-gauge will also be required and a beam compass.

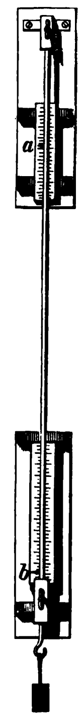


Fig. 84.

Method.—Mark about six places at equal distances on the india-rubber with ink, and measure the diameter of these by means of the wire-gauge. Take readings of the position of the needles on the scales, utilising the reflexions to avoid parallax. Now add successive weights and repeat the measurements. In using the wire-gauge the teeth must not be closer than will enable them just to touch the india-rubber, for there must be no compression of substance.

The observer will not fail to notice that after an addition of weight the stretching will continue for some time; readings should not therefore be immediately made. When the greatest weight has been added, it will be advisable to repeat the observations in a contrary order. The distance between the two scales should be determined by means of the beam compass.

Since it is the proportional compression or extension per unit of breadth or length that is required, the observed values of compression or extension must be divided by the whole breadth or length, and then the lateral compression per unit of breadth divided by the longitudinal extension per unit of length will give us the required ratio.

Having obtained this ratio by experiments of this nature, we may compare our values with what we should have obtained under the hypothesis that india-rubber is absolutely incompressible in volume. Let us therefore consider a cube of this substance of unit length and unit cross-sec-

tion. Its volume will be unity. Now, let it receive an extension of length $= \alpha$, and a diminution in lateral diameter $= \beta$: its volume will now be $(1-\beta)^2\ (1+\alpha) = 1$ as before, since the substance is supposed to be incapable of experiencing changes in volume. Hence

$$\sigma = \frac{\beta}{\alpha} = \frac{1}{\alpha}\left(1 - \sqrt{\frac{1}{1+\alpha}}\right) \quad . \quad . \quad (1)$$

From this formula it will be seen that as α increases σ diminishes. The numerical relation between α and σ is exhibited in the following table:—

$\alpha =$	4·0	3·0	2·0	1·0	0·5	·03	·001
$\sigma =$	·112	·167	·211	·293	·367	·489	·4996.

So that in the limit where α is very small σ is equal to ·5.

Conversely it may be seen that when a value very nearly $= \frac{1}{2}$ is given to σ in the formula (Art. 118)

$$\sigma = \frac{3k - 2n}{2(3k+n)},$$

the value of k, or the coefficient of cubical compression compared with that of n, or the coefficient of simple rigidity, is very great. It will be seen from the following that this is the case for india-rubber.

Example.—Successive pound weights were added, and from the mean measurements α and β were calculated from successive pairs of observations, and the value of σ was thus obtained by division. The following were the results:—

Experiment.	Load.	Length in millimètres.	Diameter in millimètres.	α.	β.	σ.
1	Clamp alone.	909·5	12·38			
2	,, +1 lb.	934·4	12·192	·02738	·01519	·55
3	,, +2 lbs.	961·4	12·025	·0289	·01369	·47
4	,, +3 lbs.	989·2	11·847	·02689	·01480	·51
5	,, +4 lbs.	1023·5	11·654	·03468	·01629	·46
6	,, +5 lbs.	1059	11·435	·0347	·01878	·53
7	,, +6 lbs.	1096·5	11·237	·0354	·01731	·48
8	,, +7 lbs.	1137·8	11·034	·0377	·01807	·47
9	,, +8 lbs.	1182·4	10·821	·0392	·0193	·49

The ratio is thus about ·5 for an extension produced by 1 lb., a high degree of accuracy being impossible in an experiment of this nature.

By the application of the formula (1) just given to any pair of observations, the value of σ may be found by calculation. Take, for instance, experiments 1 and 9; we find from these

α.	β.	σ by experiment.	σ by calculation.
·3001	·1259	·419	·414

132. The subject of elasticity presents so many interesting problems suitable for the student that a long list of these might be compiled. The following are examples:—

Elastic After-effect.—On applying a torsion force to a wire the whole effect does not immediately take place, and on the removal of the force the wire will take time to recover its original state. See "Kohlrausch," Poggendorff's *Annalen*, 1863, 1866, 1876, and 1877.

Decay of Oscillations of Torsion.—Find the law of decay, and compare the rate under different conditions. See Sir W. Thomson, *Pro. R. S.*, 1865.

Bending of Beams.—Repeat some of the experiments of Kupffer, using his method as described in his great work, *Recherches Experimentales sur l'Élasticité des Métaux.*

Spiral Springs as illustrating the theory of elasticity. See Professors Ayrton and Perry, *Pro. R. S.*, 1884. They show that a spiral spring affords a ready method of finding Poisson's ratio.

II.—Tenacity.

133. The tenacity of a body is represented by the greatest longitudinal stress per unit of cross-section that it can bear without rupture. Thus if a body have a cross-section of a square centimètres, and breaks with a weight of P grammes—that is to say, of Pg dynes, g being

the local value of gravity—then the tenacity T is expressed thus in the C. G. S. system—

$$T=\frac{Pg}{a}.$$

The tenacity of a body varies greatly with its condition as to hardness, temperature, etc. It also varies very considerably with the method of applying the longitudinal stress, for if applied suddenly the value is always greater than if applied gradually.

In the following table are given the results obtained by several experimenters for some of the common metals. In columns II. and III. the tenacity is expressed in kilogrammes per square millimètre, these units being convenient for practical work. An inspection of this table will show the extent of variation of tenacity.

TABLE U.

TENACITY OF METALS.

	I. Dynes per square cm.	II. Kilos. per sq. mm.		III. Kilos. per sq. mm.			
				Drawn Wire.		Annealed Wire.	
		Minimum.	Maximum.	Slowly Broken.	Suddenly Broken.	Slowly Broken.	Suddenly Broken.
Lead .	$2{\cdot}28 \times 10^8$	1·674	2·012	2·07	2·36	1·80	2·04
Tin .	$3{\cdot}17 \times 10^8$	2·584	4·309	2·45	2·97	1·70	3·60
Zinc .	$5{\cdot}17 \times 10^8$	...	16·328	12·80	15·77	...	14·40
Copper	$4{\cdot}14 \times 10^9$	16·380	53·252	40·30	41·00	30·54	31·82
Iron .	$5{\cdot}83 \times 10^9$	34·310	131·092	61·10	63·80	46·88	50·25
Steel .	$7{\cdot}93 \times 10^9$	61·283	121·429	70·00	92·50	40·00	53·90

I. Rankine, calculated by Everett.
II. Frankenheim.
III. Wertheim.
For brass wire 34 kilos. per sq. mm. is about the average.

For physical operations it is, as a rule, only necessary to know the tenacity of a metal in the form of wire. We proceed to a simple method applicable to this case.

Lesson XLVI.—Tenacity of Wires.

134. *Exercise.*—To determine the tenacity of specimens of copper and iron wire.

Apparatus.—One end of a selected portion of the wire about $\frac{1}{3}$ mètre long is secured by a clamp c' that is supported by a strong bracket fixed to the wall. The lower end of the wire is also provided with a clamp c, to which, by means of a hook, is attached a cylindrical tin can A about 7 inches deep and 8 inches diameter (see Fig. 85). Below the can is placed a box B nearly full of sawdust in order to break the fall of the can when rupture of the wire takes place. The box rests on a stool s, having a movable top such as was used in the experiments on elasticity. India-rubber tubing T is arranged to convey water to the can from a neighbouring supply tap. A spring-clamp affords a ready means of stopping the flow of water when desired. There will also be required a rough balance with weights, and a wire-gauge.

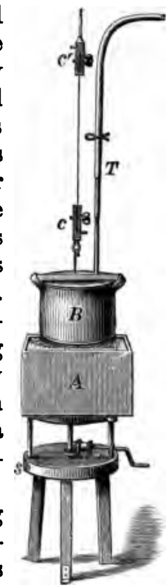


Fig. 85.

Method.—The breaking weight having been first roughly ascertained, a weight considerably under the real breaking weight is attached to or placed in the can, which at first should rest upon the sawdust. The stool is then lowered so that the can swings freely. Water is then allowed to run into the vessel until breakage takes place. The flow of water is then immediately stopped and

the can weighed. It will be found easy to repeat in this way a number of experiments, in each of which the manner of breaking is the same.

Example.—The following were the actual weights found necessary to break specimens of wire, one of copper of ·400 mm. diameter, and one of iron of ·4953 mm. diameter.

Copper 3312, 3329, 3323, 3338, 3302. Mean, 3321 grms.
Iron . 6706, 6801, 6757, 6749, 6706, 6766, 6801. Mean, 6755 grms.

We thus find—

	Mean breaking weight. Grms.	Diameter. Mm.	Cross-section. Sq. mm.	Tenacity. Kilos. sq. mm.	Tenacity. Dynes sq. cm.
Copper .	3321	·400	·1257	26·43	$2·592 \times 10^9$
Iron .	6755	·4953	·1926	35·07	$3·446 \times 10^9$

III.—CAPILLARITY.

135. The student is supposed to be familiar with the general phenomena of capillarity, such as the elevation and depression of liquids in narrow tubes, and the curvature of a liquid surface where it comes into contact with a solid. A study of these phenomena shows that every liquid behaves as if a thin film forming its external layer were in a state of tension, and exerting a constant effort to contract. This *superficial tension* is uniform in all directions, and has a constant value for a given temperature for the surface of the external layer. Further, the surface of a liquid in contact with a solid whose surface is absolutely clean makes a constant angle with the solid, known as the *angle of capillarity.* Measurements relating to capillary action usually involve the determination of the superficial tension and of the angle of capillarity. The superficial tension in the case of liquids which wet glass may be determined by aid of the following simple considerations.

136. *Theory of Capillary Tubes.*—Let ABCD (Fig. 86) be a capillary tube standing vertically with its lower end immersed in a liquid. The liquid will rise to the height

h in the capillary tube, measured from the level of the liquid at CD to the lowest portion of the meniscus AB. The superficial tension T acts in a direction inclined to the vertical at the capillary angle α; hence the force causing the liquid to rise in the tube will be the vertical component of T, or T $\cos \alpha$. If r be the radius of the tube whose section is supposed to be circular, then $2\pi r\mathrm{T} \cos \alpha$ will be the total force tending to raise the liquid. Neglecting the portion of liquid forming the meniscus, the volume of the liquid raised is $h\pi r^2$, and its weight is $h\pi r^2\rho$, where ρ is the density of the liquid. Expressed in absolute units, the force required to support this weight is $h\pi r^2\rho g$, where g is the value of gravity. We then have

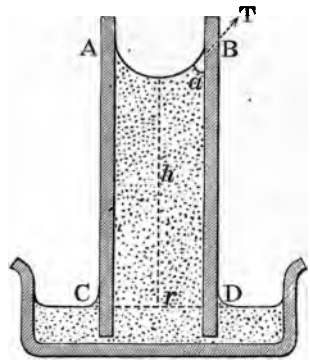


Fig. 86.

$$2\pi r\mathrm{T} \cos \alpha = h\pi r^2\rho g,$$

$$\mathrm{T} = \frac{hr\rho g}{2 \cos \alpha}.$$

If the liquid wets the glass, α is so small as to be negligible, so that in this case $\cos \alpha = 1$, and therefore

$$\mathrm{T} = \frac{hr\rho g}{2}.$$

Lesson XLVII.—Superficial Tension by Capillary Tubes.

137. *Exercise.*—To find the superficial tension of distilled water or other liquid which wets glass.

Apparatus.—A glass millimètre scale, about 30 cm. long

by 2 to 4 cm. broad, is supported by means of a wooden stand, and adjusted vertically with the help of the plumb-line *l* (Fig. 87). The lower end stands in a small shallow beaker which contains the liquid to be experimented on. Behind the beaker is placed a plane mirror MR, freely movable in any direction, and so mounted that the meniscus of the liquid in the beaker may be clearly defined, and the immersed portion of the scale illuminated by its means. Two india-rubber bands, *b b′*, are used to fix the capillary tube to the scale. There also will be required a blowpipe, soft glass tubing about 8 mm. diameter, and the microscope, with other accessories mentioned in Lesson XII.

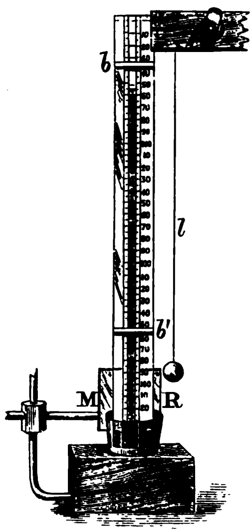


Fig. 87.

Method.—We must first prepare a number of capillary tubes. Take a piece of the glass tubing and soften it in the blowpipe, turning it round whilst being heated. When sufficiently soft remove the tubing from the flame, and draw out a capillary tube about 30 cm. (12 inches) long. Break off the capillary tubing and close both ends in a small flame, so that the inside of the tube may be kept free from dust, perfect cleanliness being essential to the success of the experiment. To prevent the outside of the tube from becoming soiled, bend one end in the form of a crook. The tube may then be hung up on a stretched string until required.

The prepared capillary tube is then fixed in position on the scale and both ends opened. The liquid which is placed in the beaker rises in the tube, and when the rise of the

liquid has ceased the tube is raised about a centimètre, so that the liquid may flow back upon a wetted surface, it being essential that the glass in the tube in the neighbourhood of the meniscus should be wetted. Readings may be taken of the level of the liquid and of the height of the meniscus with the aid of a magnifying glass. In estimating the latter the height read off should be a little greater than the bottom of the meniscus, in order to allow for the error due to the curvature of the meniscus.[1] We shall now require to know the diameter of the tube at the point to which the liquid has risen. The capillary tube must for this purpose be removed from the glass scale, and a small portion (as near as may be to the required point) broken off by help of a file with a knife-edge. The next procedure is that already described in Lesson XII.[2]

With a proper arrangement of the light no difficulty should be experienced in reading off the exact level of the fluid in the beaker, as defined by the lower boundary of the meniscus, provided that the fluid is sufficiently transparent. If this be not the case, a fine sewing-needle of measured length may be fastened by sealing-wax to the bottom of the scale, so that the point projects. By lowering and raising the scale or the level of the liquid the needle-point is arranged just to touch the liquid. A reading of the position of the upper end of the needle and of the meniscus in the capillary tube will then give the height required.

Example.—Determination of capillary constant of distilled water at 15° C., density taken as unity:—

	Radius $=r$.	Height $=h$.	Tension $=\frac{rh}{2}$.
	Cm.	Cm.	Grammes per cm.
1 . .	·0178	8·12	·0723
2 . .	·02645	5·57	·0737
3 . .	·0420	3·43	·0720

[1] Several approximate formulæ have been proposed to allow for this error. The simplest requires that $\frac{1}{3}$ of the radius of the tube should be added to the observed height.

[2] The chief details of the above lesson are due to Professor Quincke.

138. Further treatment of the subject of capillarity being beyond the scope of these lessons, the student who is desirous of following up this interesting subject should consult some of the general and special treatises, such as—

Jamin, *Cours de Physique*, vol. i., new edition.

Wullner, *Lehrbuch der Experimentalphysik*, vol. i. pp. 304-357.

Encyclopædia Britannica, article "Capillarity," by Clerk-Maxwell.

"Capillarité," by M. A. Terquem, extracted from the *Encyclopédie Chimique*.

We would further propose the following as additional exercises:—

(1.) Determine the capillary constants by the method of Quincke (Poggendorff's *Annalen*, lxxxix.). Air-bubbles in liquid confined in a glass cell and drops of mercury are directly measured by the cathetometer microscope.

(2.) Repeat the beautiful experiments of Plateau on the equilibrium of oil drops floating in a liquid of their own density, and on the shapes assumed by films. See the remarkable work of this blind philosopher, *Statique des liquides soumis aux seules Forces Moléculaires.*

(3.) Repeat the experiments of Salleron and others on the relation between the weight of a drop of a liquid and its superficial tension. Count the number of drops of different liquids delivered by a pipette with a capillary opening. Make and test different simple forms of "drop-counters." See Jamin and the references given there.

(4.) Laws relating to the formation of drops and bubbles. Fit up apparatus and repeat some of the experiments of Guthrie, *Pro. R. S.*, vol. xiii.

(5.) See also a paper by A. M. Worthington on "Pendent Drops," *Pro. R. S.*, vol. xxxii.; and one by the same author on "Impact with a Liquid Surface," *Pro. R. S.*, vol. xxxiv.

CHAPTER VII.

Determination of Atmospheric Pressure.

139. STANDARD PRESSURE.—The pressure of the atmosphere at a given place and time is usually measured by the height of pure mercury at 0° C. that it can support. Many physical operations, such as the determination of the volume of a gas or the boiling point of a liquid, being affected by the ever-varying pressure of the atmosphere, it is necessary to select a *standard pressure* for reference. The standard commonly in use and suited for most purposes is the height of 76 cm. (29·922 inches) of pure mercury at 0° C. (which is near the average height of the barometer). Since it is, as a rule, the pressures relative to this standard that are required in laboratory experiments, a simple observation of the barometric height is all that is usually necessary. But since the actual pressure in units of force varies with the value of gravity, and therefore with the position on the earth's surface, we must include in our statement of a standard pressure some selected locality. This, however, may be avoided if we express our standard in a different way. Let a dyne (see Appendix) be taken as the unit of force, then the pressure of the atmosphere will be defined as that pressure in dynes per square centimètre which is produced by the barometrical column of mercury. Thus if H centimètres be the height of the barometer at 0° C., and g the value of gravity, then the pressure expressed in this way will be 13·596 Hg dynes,

where 13·596 is the density of mercury at 0° C. If the place be Greenwich, where $g = 981{\cdot}17$, and if H be at the standard height of 76 cm., then the pressure will be 1,013,800 dynes; whereas for Paris the same barometric height would be equivalent to a pressure of 1,013,600 dynes. Seeing, therefore, that this standard height gives approximately a pressure of one million dynes or a mega[1]-dyne, it is proposed to make one **mega-dyne per square centimètre** the standard pressure. This new unit has been adopted by several modern writers, and the points in its favour are—(1) convenience of calculation; (2) scientific accuracy in stating the pressure; (3) its independence of the local value of gravity.

140. *The Barometer.*—The type of mercurial barometer most commonly employed for accurate observations is the *cistern barometer*. A good instrument of this kind consists simply of a straight tube closed at one end that has been filled with perfectly clean and dry mercury to the exclusion of all air-bubbles, and then inverted in a cistern containing mercury, the tube being of such a length that a Torricellian vacuum is produced. When constructed so that the instrument may be safely moved it is called a *portable or working barometer*; when of specially large size and intended to be permanently fixed at the place where the tube has been filled, it is then known as a *standard barometer*.

LESSON XLVIII.—**Use of Working Barometer.**

141. *Exercise.*—To practise setting the barometer and to determine its reduced height.

Apparatus.—A barometer constructed on the principle of Fortin. The peculiar feature of this barometer consists in the construction of the cistern, which is shown in section in Fig. 88. The brass cover at the top of the cistern has

[1] *Mega* as a prefix means "million."

a central aperture to admit the constricted end of the barometer tube. Below the cover is a short cylinder of glass, through which we view the mercury level. This cylinder rests upon the lower brass-protecting casing, the three parts—namely, the cover, the glass cylinder, and the lower casing—being bound together by long screws, two of which are seen in the engraving. The lower inner wall of the cistern is of boxwood, to which is fastened beneath a bag of leather. The object of this arrangement is twofold—*First*, by means of the screw seen at the bottom of the cistern the level of the mercury may be adjusted so as to stand at a constant height, indicated by a small ivory pointer; *Secondly*, when the instrument is about to be moved the screw may be turned until both the tube and the cistern are filled with mercury, and then the instrument may be laid in any position without danger of introducing air or breaking the tube by the bumping of the heavy metal against the glass. The method of fitting the lower end of the tube to the top of the cistern will be seen in the engraving; at the top of the cistern there is a lining of boxwood, having attached to it a piece of *chamois* leather, so arranged as to entirely prevent the escape of mercury, but at the same time to allow free communication to the outside air through the porous leather.

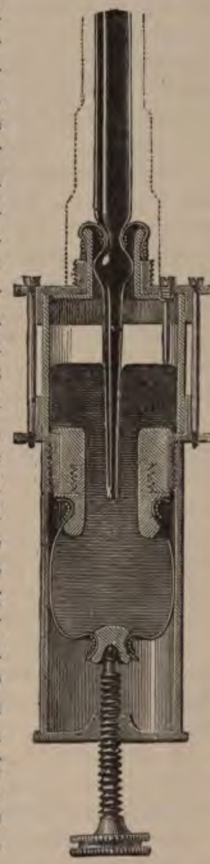


Fig. 88.
CISTERN OF FORTIN'S BAROMETER.

The whole length of the glass tube

is enclosed within a brass tube for protection, part of this being represented by the dotted lines of the figure. Near the upper end of the tube two rectangular spaces, front and back, permit an observation of the height of the mercury, whilst the front edges carry the graduations (see Fig. 5). There is also a short metal cylinder, the front portion forming the Vernier, which is movable by means of a rack and pinion within the two rectangular spaces.

The barometer must be fixed with its scale about five feet high, in a good light, and in a position as little as possible exposed to fluctuations of temperature. It is also very essential that the instrument should be fixed vertically. To assist in securing this the barometer is suspended by a ring at the top from a hook, so that it is free to assume a vertical position. When so suspended its lower extremity should be clamped by the three screws that are provided for the purpose.

A thermometer attached to the brass casing gives the temperature of the instrument.

Method of Observation.—The operations should be performed in the following order:—

1. Read the temperature at once, for otherwise the vicinity of the body of the observer may vitiate the observation. If the errors of the thermometer are known the corrected temperature should be recorded.

2. Tap the barometer in the vicinity of the upper level of the mercury: this enables the mercury to overcome any slight impediments and the meniscus to assume its proper shape.

3. Adjust the level of the mercury in the cistern. To do this, lower, if necessary, and then raise the cistern screw until the level of the mercurial surface just touches the tip of the ivory pointer. This will be indicated by a slight depression and irregularity of the surface. With the aid of a magnifying glass and a proper arrangement of the light a very exact setting may be obtained, providing that

the surface of the mercury is quite clean. But when the surface of the mercury becomes soiled, which often occurs, an exact setting is difficult, especially when dirt collects opposite the end of the pointer. In this case a clean surface may be sometimes procured by raising the mercury rather high in the cistern, and then lowering and shaking the cistern, this operation being repeated several times if it should be necessary.

4. Set the Vernier. The bottom of the Vernier must be brought so as to be an apparent tangent to the convex surface of the meniscus, and in this case the background (a tablet of white enamelled glass is best) will be visible at the edges only. To avoid parallax advantage must be taken of the movable tube on which the Vernier is graduated. The back lower edge of this ought to coincide with the front lower edge when viewed from the proper position. When thus set, if the eye be moved slightly up and down, no line of light should appear in the middle, and we are then sure that the line of sight is horizontal.

5. Read one or both the Verniers as described in Lesson III.

Reduction of Barometer Observations.—The direct reading of the barometer represents the distance from the tip of the ivory pointer to the top of the mercury meniscus. It requires several corrections, which we shall now consider.

I. *Correction for Temperature.*—Since the density of the mercury and the length of the divisions of the scale both depend upon temperature, it is necessary to refer the observed barometric height to the respective standard temperatures of the mercury and the scale. Let α denote the cubical expansion of mercury, and β the linear expansion of the material of the scale for one degree of temperature. Call H the observed height of the barometer at temperature T°, and let H_0 be the corrected height. We shall first of all suppose that the standard temperature of both mercury

and scale is the same, namely, 0° C., and that the system is the metrical one.

A. *Metrical System.*—Since at the standard temperature, 0° C., the H divisions were of correct length, at T° they are really of the length $H(1+\beta T)$.

But the height of the mercury will be inversely as its density, hence

$$\frac{H_0}{H(1+\beta T)} = \frac{1}{1+\alpha T},$$

or

$$H_0 = H\,\frac{1+\beta T}{1+\alpha T} = H(1-(\alpha-\beta)T) \text{ nearly.}$$

The value of β will depend upon the nature of the scale employed; let us suppose it to be of brass.

Hence for 1° C. $\alpha = {\cdot}00018018$ and $\beta = {\cdot}00001878$, so that $\alpha - \beta = {\cdot}0001614$. Therefore for all temperatures above 0° C. the correction is $-{\cdot}0001614HT$.

The following table gives the value of this correction :—

TABLE V.

REDUCTION OF METRICAL SCALE OF BAROMETER WITH BRASS SCALE TO 0° C.

°C.	640	650	660	670	680	690	700	710	720	730	740	750	760	770	780
10	1·03	1·05	1·07	1·08	1·10	1·11	1·13	1·15	1·16	1·18	1·19	1·21	1·23	1·24	1·26
11	1·14	1·15	1·17	1·19	1·21	1·23	1·24	1·26	1·28	1·30	1·31	1·33	1·35	1·37	1·38
12	1·24	1·26	1·28	1·30	1·32	1·34	1·36	1·38	1·39	1·41	1·43	1·45	1·47	1·49	1·51
13	1·34	1·36	1·38	1·41	1·43	1·45	1·47	1·49	1·51	1·53	1·55	1·57	1·59	1·61	1·64
14	1·45	1·47	1·49	1·51	1·54	1·56	1·58	1·60	1·62	1·65	1·67	1·69	1·72	1·74	1·76
15	1·55	1·57	1·60	1·62	1·65	1·67	1·69	1·72	1·74	1·77	1·79	1·82	1·84	1·86	1·89
16	1·65	1·68	1·70	1·73	1·76	1·78	1·81	1·83	1·86	1·89	1·91	1·94	1·96	1·99	2·01
17	1·76	1·78	1·81	1·84	1·87	1·89	1·92	1·95	1·98	2·00	2·03	2·06	2·09	2·11	2·14
18	1·86	1·89	1·92	1·95	1·98	2·00	2·03	2·06	2·09	2·12	2·15	2·18	2·21	2·24	2·27
19	1·96	1·99	2·02	2·05	2·09	2·12	2·15	2·18	2·21	2·24	2·27	2·30	2·33	2·36	2·39
20	2·07	2·10	2·13	2·16	2·20	2·23	2·26	2·29	2·32	2·36	2·39	2·42	2·45	2·49	2·52

The correction is in millimètres and must be subtracted.

B. *English Units.*—Using Fahrenheit's thermometer, and assuming the brass scale to be correct at 62° Fahr., the above formula will be required to be somewhat modified. At T° the observed H inches are really

$$H\ \{1+\beta'(T-62)\}$$

inches, and the density of the mercury at T° is equal to that at 32° Fahr. multiplied by the factor

$$\frac{1}{1+\alpha'(T-32)}.$$

Hence

$$H_0=H\times\frac{1+\beta'(T-62)}{1+\alpha'(T-32)},$$

or

$$H_0=H\ \{1-\alpha'(T-32)+\beta'(T-62)\}\text{ nearly,}$$

where

$$\alpha'=\cdot0001001,\ \beta'=\cdot00001043.$$

By inserting these values we obtain the following more convenient and sufficiently accurate formula—

$$H_0=H-H\,\frac{(\cdot09\ T-2\cdot56)}{1000}.$$

II. *Capillarity and Index Errors.*—The effect of capillary action is to cause a depression of the mercury. Tables have been compiled showing the correction to be applied with different diameters of tube and different forms of meniscus (the meniscus is more convex when the barometer is rising than when falling). Since, however, the correction is somewhat uncertain, being greatly influenced by the state of cleanliness of the mercury, it is better to eliminate it as far as possible by having a tube of sufficiently great diameter; thus, with a tube of ·75-inch diameter, the correction is less than ·001 inch. This error, as well as those due to incorrect placing of the ivory pointer, errors of graduation and of the attached thermometer, are best

obtained by comparison with standard instruments. This comparison is undertaken at the Kew Observatory, from which a "certificate of examination" may be obtained. The following statement will show the nature of this:—

Large Portable Standard No. 936 compared with the Standard Barometer of the Kew Observatory.

Correction (including capillary action) English scale – ·001 inch.
,, ,, ,, Metrical scale – ·100 millimètre.

CORRECTION TO ATTACHED THERMOMETER.

At 32°	At 42°	At 52°	At 62°	At 72°	At 82°	At 92°
–0°·1	–0°·1	–0°·0	–0°·0	+0°·1	+0°·1	–0°·0

Note.—When the sign of the correction is + the quantity is to be added to the observed scale reading, and when – to be subtracted from it.

Example.—The mercury was adjusted, the Vernier set, and the reading taken of the barometer ten times successively. The following were the results:—

29·550, 29·555, 29·550, 29·554, 29·548
29·544, 29·544, 29·545, 29·546, 29·550[1] } Mean, 29·549 inches.

Temperature at commencement, 60°·5; at end, 61°·0. Mean, 60°·75 Fahr.
Correction for temperature by above formula – ·086 inches.
Correction for capillarity, etc., by Kew certificate – ·001 inch.
Reduced reading at freezing point, 29·549 – ·086 – ·001 = 29·462 inches.

[1] The probable error of a single reading obtained by the method given in the Appendix was found to be ·0024 inches, and the probable error of the mean of the series was ·00076 inches, assuming the barometer to have been constant during the time of the observation. These readings were taken by an observer who was not specially practised. They probably represent the accuracy ordinarily obtainable. In estimating

LESSON XLIX.—Comparison of Standard with Working Barometer.

142. Having now described a working barometer, and the method of observing with it, we proceed to describe an instrument the object of which is to test the accuracy of a working barometer.

The standard barometer in the Owens College Physical

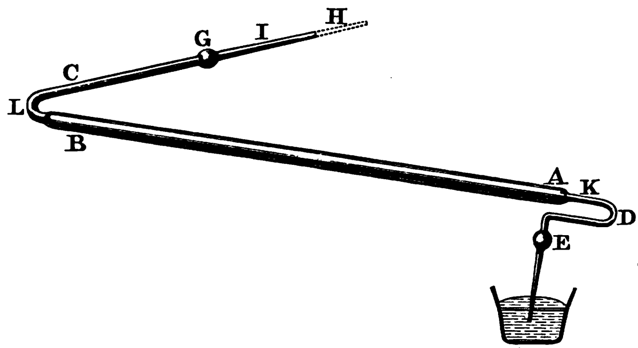


Fig. 89.

Laboratory has been filled after the manner invented by the late John Welsh of the Kew Observatory, and described by him in the *Transactions of the Royal Society* for 1856. A tube of glass of about 1¼ inch internal diameter, having a capillary tube AKDE attached to it as in Fig. 89, is cleansed by sponging with whiting and spirits of wine.

the value of the result the following remarks of Regnault should be remembered:—"The pressure of the atmosphere varies incessantly, but this variation is not immediately indicated by the barometer but by the change of form of the meniscus; and the variations of height do not take place in a continuous manner, but rather by fits and starts. To avoid this, the barometer must be shaken and caused to oscillate up and down, but it is not possible to eliminate altogether this source of error."

It has then another appendage CGIH attached to it, and the whole is finally exhausted, slowly, with a good air-pump, and at the same time the whole tube is well heated. The capillary tube is then sealed off whilst the instrument is in this state. After the tube has cooled, it is placed with the end of DE in boiled and purified mercury, and this end being then broken off the mercury slowly fills the tube. When sufficient mercury has entered, the orifice at the end of DE is sealed by a blowpipe. If the tube be now placed erect the mercury will separate at D, leaving the space from D to the end filled with mercury, but that from D to A empty. The tube is now sealed at K, removing the portion DE. Finally, the lower end of the tube being placed in its proper cistern of mercury, is broken at C, leaving about an inch of siphon.[1]

Fig. 90.—STANDARD BAROMETER.

The barometer, constructed in this manner, is afterwards mounted with an iron framework, and the whole supported on a solid block of masonry, appearing as in Fig. 90. There is a sufficient Torricellian vacuum above

[1] Our laboratory standard tube was made and filled by Mr. John Hicks of London.

the mercury; so that even if in the course of time certain minute air-bubbles should enter this vacuum, they would not be able to produce a perceptible pressure.

The instrument, cistern and all, is capable of being turned round on an axis, which must be strictly vertical. In order to procure this verticality there are two iron pointers attached to the instrument. These are, by means of a fine screw motion, made to touch the surface of the mercury in the cistern by the method of reflexion already described, and the axis is adjusted by means of certain screws which work on the iron bed-plate until this contact of the two pointers with the mercurial surface is perfect in all azimuths. In this case the line joining the extremities of the pointers must be horizontal in all positions, and hence the axis must be vertical. Having made this adjustment, the pointers being in accurate contact with the mercury of the cistern, the observation consists in measuring the vertical distance between the top of the convex surface of the mercury and a mark made on one of the pointers, the distance of which from the lower extremity of the pointer has been previously ascertained. The whole distance from the top of the convex mercurial surface to the surface of the mercury in the cistern will thus be obtained by adding together the two measurements—that is to say, adding to the distance between the convex surface and the mark on the pointer the previously ascertained distance between this mark and the extremity of the pointer.

There is one point which requires attention in taking an observation of the height of the mercurial column of such a barometer. It is to prevent reflected light from giving us a false position for the top of the convex surface. This surface is viewed by the telescope of the cathetometer, and the horizontal cross-wire of this telescope is made to lie in contact with it. If, however, the convex surface be lighted up with reflected light, the position of this contact may prove to be inaccurate. To prevent this a thick paper

cylinder of about 3 inches in height is made to surround the barometric tube, extending upwards from within a tenth of an inch or so of the mercurial surface, and a thin sheet of paper is likewise placed behind this surface in a vertical position; no stray light is thus allowed which can be capable of producing a false impression. If the illumination be not sufficient, a gas flame may be placed behind the vertical sheet of paper. The contact is then made between the wire of the telescope and the surface.

There are two ways in which such an observation may be taken. In the first of these the vertical height may be read off on the scale of the cathetometer itself, and in the second this vertical height may be read off on a scale placed alongside the standard barometer, a telescope being used for the purpose of reading.

We shall now proceed to describe observations made by both these methods, the object being to take simultaneous readings of the standard and working barometers, with the view of determining the error of the latter.

Method I.—Comparison made by Cathetometer Scale.—The following comparison by Morisabro Hiraoka was made in 1878 at our Physical Laboratory. The preliminary determinations were—

(1.) Comparison of the cathetometer scale with the standard yard, placed at the same distance as the barometer. From the mean of seventeen readings it was found that 1000 divisions of the cathetometer scale is equal to 999·88 mm. with a probable error of ±·01 mm.

(2.) Comparison of the dividing machine with the standard yard, and determination of the length of the barometer screw-head pointer by the former. From the mean of ten readings its length was found to be = 111·340 mm. with a probable error of ±·001 mm.

(3.) Comparison of the English and metrical scales of

the working barometer. From the mean of a number of comparisons at heights between 28 and 31 inches it was found that ·052 has to be taken from the reading of the metrical scale in order to make it agree with the scale of inches. (The Kew certificate makes the difference to be ·050, so that the two determinations coincide very well together.)

Having made these preliminary determinations, the next point was to make an actual comparison between the two barometers. This was done in the following order—

(A.) A reading was taken of the upper surface of mercury of the standard barometer.
(B.) A reading was taken of the working barometer.
(C.) The screw-head of the standard was adjusted.
(D.) A was repeated.
(E.) B was repeated.
(F.) Temperatures were noted.
(G.) The height of the centre of the cross or mark on the screw-head was read twice.

Twenty-two individual observations were made, and the following results were obtained after all corrections had been made, amongst which was the correction of the metrical scale of the working barometer, in order to make it comparable with the scale of inches assumed to be correct—

	Millimètres.
Mean of standard barometer . .	772·077
,, working ,, . .	772·103
Working barometer reads too high	0·026, or 0·0010 inches.

In these twenty-two comparisons the mean difference of an individual comparison from the mean of the whole was ·086 millimètres, which is a somewhat large result.

Finally, adding a small correction due to the difference in height of the cisterns of the two barometers, the final

result was that the English scale of the working barometer read ·0009 inches too high.

This is in good accordance with the Kew verification, derived by a comparison with the Kew standard barometer, which reports our working standard to be correct to the third place of decimals in inches. The mean error of a single determination, as already mentioned, was somewhat large. This was no doubt due in part to the too great distance (about 12 feet) between the standard barometer and the cathetometer. This has since been altered, the present distance being about 4 feet.

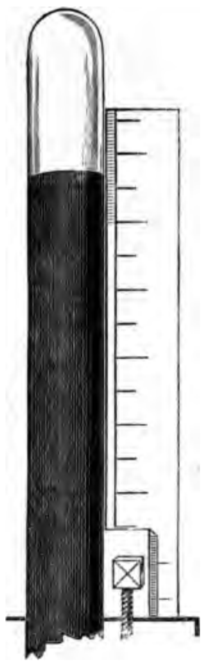


Fig. 91.

Method II.—Comparison by Reading Telescope.—This method was designed to avoid the use of the cathetometer —an instrument which, as we have already pointed out, does not give satisfactory results. It consists simply in the use of a fixed scale and a reading telescope. A brass scale was fixed up by the side of the barometer tube, the bevelled edge of the scale being quite close to the tube, except for about 3 inches from the bottom, where it was necessary to make room for the head of the adjusting screw at the top of the cistern. For this purpose a rectangular portion, 3 inches by 1 inch, was cut out of the bottom of the scale, and the bevelling continued along the 3 inches of edge, which was now removed 1 inch away from the tube (see Fig. 91). The scale was graduated in metrical measure, the division marks being half a décimètre apart in the centre of the scale; but in the working por-

tions—namely, within the possible range of the mercury meniscus and of the head of the adjusting screw—the divisions were half-millimètres, the lines being very fine. The scale was viewed by the cathetometer telescope (any other properly mounted telescope would have done as well), the position on the divided scale of the mercury meniscus and of a reference line on the head of the adjusting screw being directly read off, estimation being made to the one-tenth part of a division, or ·05 mm. Using this method, Mr. Isidore Kavanagh has been enabled to compare our working barometer with the standard.

CHAPTER VIII.

Time, Gravitation, Moments of Inertia.

I.—Measurement of Time.

143. Standard of Time.—Time may be measured by the help of any regularly-recurring event. In seeking for a standard unit of time we select those events which, after the most rigorous tests, have been proved to take place with the greatest possible regularity.

The time taken by our earth to perform one revolution round its axis may be practically regarded as invariable. No doubt, millions of years ago this time may have been different from what it is now, and millions of years after it may again be different; but the time that will be required for this performance to-morrow will most certainly be the same as that required for it to-day.

While the time occupied by one revolution of our earth is our standard of ultimate reference as regards time, yet it does not form our working standard. A well-constructed timekeeper, either clock or chronometer, is necessary to our requirements. Let us imagine, to begin with, that we have a good sidereal clock, and that at the present moment precisely 24 hours 0 minutes 0 seconds of this clock denote the time occupied by the earth in revolving on its axis. Six months after this we make another comparison, and find that the time of one revolution of the earth is no longer 24 hours,

but perhaps 24 hours and 1 second. The question then arises, whether it is the earth's time of rotation or the clock's rate that has changed. This is answered by the astronomer and physicist, who have jointly brought forward an enormous mass of evidence to show that the earth's time remains constant, while that of the clock must have altered. We shall now indicate in what manner we make a comparison between our sidereal clock and the earth.

Let a telescope with a vertical cross-wire be mounted so that its optical axis may be capable of moving exactly in the plane of the meridian—that is to say, in an imaginary plane passing through the north and south points, and also through the zenith of the observer—and let the passage of any particular *fixed star* over the vertical cross-wire be observed from time to time. It will be found that the "transits" occur regularly without the slightest apparent variation. The interval of time that elapses between two transits is called a **sidereal day**, which is the unit used by astronomers. A sidereal day really expresses the time that the earth takes to turn once round its axis, for the star may be regarded as at an infinite distance and immovable. A sidereal clock expresses this day, and divides it into 24 hours, each hour into 60 minutes, and each minute into 60 seconds of *sidereal time.*

If the transits of the centre of the sun be observed, then the interval of time between successive transits will be called **solar days.** These solar days will not be found to be of equal length, so that it would not be advantageous to choose an actual solar day as a unit of time. Instead, however, a **mean solar day**—a perfectly arbitrary unit, obtained by averaging the length of the solar days in a year—has become the accepted unit for all civil and most scientific purposes. A mean solar day would thus be produced by an imaginary sun, called the **mean sun**, travelling at a uniform rate with such a speed that it would in a year complete the same journey as the real sun. There

will thus evidently be a difference in the time of occurrence of *noon*, as given by the transit of the real and of the imaginary sun across the meridian of the place. This difference is called the **equation of time**, and a knowledge of this quantity is necessary in order to determine mean solar time by astronomical means.

A mean solar day is divided into 24 hours, an hour into 60 minutes, and a minute into 60 seconds, the subdivision of the seconds being written decimally. These will all be subdivisions of mean solar time, and must be distinguished from the similar subdivisions of sidereal time.

Again, all of these subdivisions of time must be distinguished from minutes and seconds of angular measure. To do this we write the smaller units of time thus—$\overset{m.}{16}\ \overset{s.}{5{\cdot}3}$. A mean time second is therefore $\frac{1}{24 \times 60 \times 60} = \frac{1}{86,400}$ part of a mean solar day. As thus defined, the **second** is the standard of time used in physical measurements. This mean time second may be shown to be the $\frac{1}{86,164{\cdot}09}$ part of a sidereal day.

Since the sun crosses the meridians of two places at different times, it will be necessary, in stating the absolute time of any event, to specify some meridian. In Britain such time is defined with reference to the meridian of Greenwich, which has lately been accepted as a **universal meridian.**

In physical operations the absolute time is very rarely required, a knowledge of the difference in time between two events being usually sufficient; so that in the laboratory we need only be provided with some mechanism such as a clock or chronometer, which records mean time seconds.

144. *The Clock.*—Let the student examine a good clock with pendulum. Moving with the pendulum there is a *dead-beat escapement* with its two teeth, which, by a step-by-step movement, causes a wheel with acute teeth (the *escapement-wheel*) to rotate. The action of the escape-

ment and its wheel deserves careful study. These have been so admirably contrived by their inventor, the celebrated Graham, that the natural time of oscillation of the pendulum is not interfered with, the escapement-wheel giving the impulse necessary to maintain the motion of the pendulum precisely at the time when the pendulum is crossing its position of rest. This impulse is originally due to the clock weight, whose gradual fall is the means of imparting energy to the clock-works. The impulse so received by the pendulum is very small, and need only be sufficient to allow for loss of energy by friction, etc. The method of registering the number of seconds, minutes, and hours does not need special description here.

The clock, to keep accurate time, must have a *compensation* pendulum, made so that the effect of changes of temperature in altering the length of the pendulum, and thus changing its time of vibration, may be counteracted. Details of the various methods of compensation employed will be found in any treatise on heat or general physics.

145. *The Chronometer.*—In timekeepers which are required to be portable the pendulum is not admissible. The *isochronous* system here made use of will be understood at once if the student will examine an ordinary watch. A wheel called the *balance-wheel*, revolving on small steel pivots that work in jewelled holes, has attached to it a flat spiral steel spring, called the *hairspring*. Under the action of this spring the wheel vibrates in exactly equal times. The balance-wheel may swing through a large angle without loss of isochronism, and in this respect it has a great advantage over a clock pendulum, in which the amplitude of swing must be small, as will presently be more fully explained. The balance-wheel is kept in oscillation by the impulse from the escapement-wheel, whose rate of rotation the balance-wheel controls. The driving power is a coiled spiral spring, called the *mainspring*. As this spring becomes

mercury clock. It is evident that when the interval of time to be measured is very small, any of these arrangements will be unsuitable. In this case the isochronous property of a vibrating-rod or tuning-fork may be used, as in the **Chronograph of Hipp**, in which a reed, vibrating 1000 times in a second, replaces the pendulum of a clock; the instrument will therefore tick 1000 times a second, and will record with accuracy small intervals of time. To vibrating-reeds or tuning-forks a number of other similar time instruments owe their construction, whilst, by calling in the aid of electricity, *electro-chronographs* of great convenience and accuracy have been produced. Some simple forms of these instruments we shall describe more fully in the course of this chapter.

II.—GRAVITATION.

149. We shall now apply methods of time measurement to a study of gravitation.

The Pendulum.—We shall begin by determining the law of motion of a body constrained to oscillate about a fixed axis. A body compelled to move in this way is called a pendulum, and forms an instrument of great physical importance and utility. Investigations regarding pendulums are greatly simplified by considering an ideal simple pendulum consisting of a heavy particle attached to a weightless inextensible thread. In practice this condition is imitated, as far as possible, by using a small heavy ball attached to a very fine wire.

LESSON L.—Law of Pendulums.

150. *Exercise.*—To find the relation between the time of a *single oscillation*[1] of a simple pendulum (oscillating under the action of gravity) and its length.

[1] We shall always consider an oscillation or vibration to consist of a *single swing.*

Apparatus.—A ball of brass about 1 inch (25 mm.) diameter is suspended by a fine wire about a mètre long. The upper end of the wire is provided with a small polished steel knife-edge, which rests on polished steel bearings. The method of fitting up this apparatus is described in detail in the Appendix. For measuring the length of the pendulum the apparatus of Fig. 92 must be provided. It consists of a long rod of wood about 1 inch square. At *a* is a knife-edge, and at B is a movable portion, having a clamp screw at *b* so as to allow the rod to be varied in length from $\frac{3}{4}$ to $1\frac{1}{4}$ mètre. The end of the rod is provided with a screw S for fine adjustment; this terminates in a brass disc *d* slightly greater than the diameter of the rod. In conjunction with the measuring rod there is employed the small brass stand C, about 3 inches high, into which a movable upper portion terminating in a knob *p* is made to screw. We shall also require a pair of callipers and a millimètre scale. For the measurement of time, a clock, chronometer, or a stop-watch will likewise be needed.

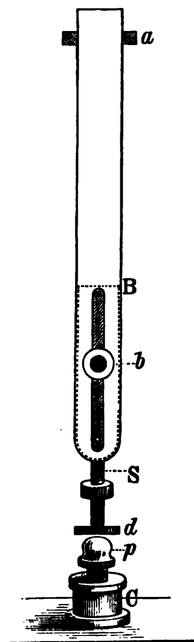


Fig. 92.

Method.—Set the pendulum swinging through a small arc, and observe the time at which it crosses the middle of its path when passing, say, from right to left of the observer. Call this the 0th oscillation, then count the number of passages until the 100th—a passage again from right to left—is reached, and once more observe the time. Divide the whole time by 100 to obtain the time of a single oscillation.

Now place on a support below the pendulum bob the stand C, and screw up its knob until the lower surface of the pendulum bob, as it swings to and fro, just touches the upper surface of the knob *p*. This adjustment—which may be made with the greatest nicety—having been completed, remove the pendulum, taking care meanwhile that the stand C remains undisturbed, and substitute for the pendulum the measuring rod. Adjust the length of the rod by means of the sliding piece and the adjusting screw S until the disc *d* just touches the convex end of *p*. We have thus transferred the length of the pendulum into a condition in which the measurement of its length will be easy, and can be made with considerable accuracy if required. For our present purpose it will be sufficient if the length from the knife-edge to the end of the brass disc *d* be obtained by applying the rod to a millimètre scale. Find the diameter of the bob of the pendulum by callipers, and then obtain the length from the knife-edge to the centre of the bob. Let the length of the pendulum be altered, and repeat the process of measuring its time of oscillation and its length. Do this several times.

From the results obtained it will be perceived that as we shorten our pendulum the time of an oscillation becomes less. Now, what is the law that connects the length of the pendulum with the time of an oscillation? The appearance of the numbers leads us to suspect that the squares of the times vary as the lengths; so that if t = time of an oscillation, and l = length of the pendulum from knife-edge to the centre of the bob, the $\frac{t^2}{l}$ should give us a constant number. Hence, when we find the values of $\frac{t^2}{l}$ for each experiment, we obtain a series of numbers which should agree tolerably well together, if our hypothesis be correct.

Example.—

Experiment.	Time in Seconds.	Length in Millimètres.	$\frac{t^2}{l}$.
1	1·05	1132·5	·0009735
2	1·025	1072	·0009800
3	·995	1015·5	·000975
4	·985	986	·000984

151. *Law of Ideal Simple Pendulum.*—The law which we have proved experimentally might have been deduced, *a priori*, from simple considerations relating to the laws of motion. A demonstration which the student will find in all works on mechanics proves the truth of the following formula—

$$t=\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where g is the intensity of gravity. This expression assumes that the amplitude of the swing is very small. If the pendulum be making swings of not more than 2° or 3° on either side of the vertical, the error introduced by employing this formula is so small as to be negligible; when the deviation amounts to 5°, the error would only amount to $\frac{1}{2000}$ part of the time of the swing. A more correct but still approximate formula gives

$$t=\pi\left(1+\frac{1}{4}\sin^2\frac{A}{2}\right)\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad (2)$$

where A is the mean angle which the pendulum makes with the vertical.

152. *Correction to Simple Pendulum.*—If the length of a pendulum, consisting of a very small knife-edge, a fine wire,

and a small heavy spherical bob, measured from the knife-edge to the centre of the sphere, be l, and r be the radius of the sphere, the system will actually swing (neglecting the error due to the wire and knife-edge, and considering that due to the sphere alone) as an ideal simple pendulum of length $l+\frac{2r^2}{5l}$. The student will understand the reason of this correction when moments of inertia are considered.

153. *Determination of Gravity by the Simple Pendulum.*—From the last two articles we obtain the formula—

$$g=\frac{\pi^2}{t^2}\left(l+\frac{2r^2}{5l}\right)\left(1+\frac{1}{4}\sin^2\frac{A}{2}\right)^2 \quad . \quad . \quad . \quad (3)$$

This result requires further correction when the pendulum is swinging in air. If P be the weight of the pendulum *in vacuo*, and p its loss of weight in air, the actual force acting on the pendulum is $P-p$. So that if we call G the value of gravity *in vacuo*, we shall have

$$\frac{G}{g}=\frac{P}{P-p},$$

or

$$G=g\frac{P}{P-p} \quad . \quad . \quad . \quad . \quad (4)$$

Several other corrections must be considered in an exact determination. These are due (1) to resistance of the air; (2) to viscosity of the air; (3) to air carried by the pendulum in its motion; (4) to loss of energy due to the supports and disturbance caused by neighbouring vibrations.

LESSON LI.—Intensity of Gravity by Borda's Method.

154. *Exercise.*—To apply the pendulum to find the value of gravity.

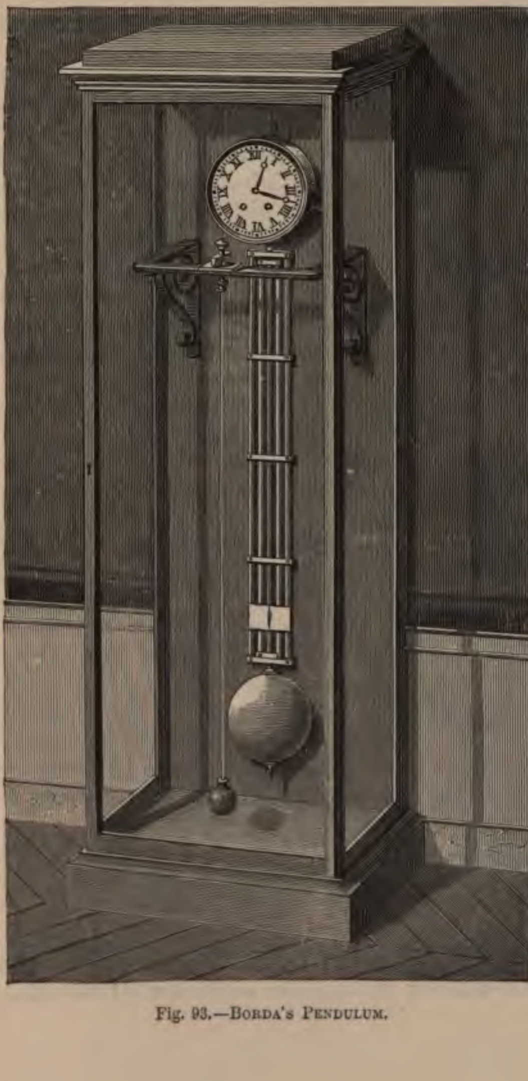


Fig. 93.—Borda's Pendulum.

Apparatus.—Fig. 93 shows the arrangement of the apparatus which, with some little modifications, we shall adopt. A simple pendulum supported by its knife-edge rests on smooth bearings in front of a clock beating seconds. The simple pendulum is of such a length that it has nearly the same time of oscillation as the clock pendulum. Attach a light mirror to the lower end of the bob of the clock pendulum, with its plane so inclined that light from a lamp placed at the side of the clock may be reflected towards the simple pendulum. From the bob of the latter hangs a light paper tail, so that the light reflected from the mirror is intercepted when the pendulums are vertical. The flashes of light are observed by a telescope of low magnifying power, placed about 2 mètres away. The details of the pendulum are seen in Fig. 94. By the aid of two adjusting screws placed above and below the knife-edge its time of vibration is adjusted so as to vibrate nearly in a second; in this way any correction due to the knife-edge is avoided. The apparatus of the previous Lesson, for measuring the length of the pendulum, will also be required.

Method.—Set the pendulum vibrating through a small arc. To do this, without at the same time causing the pendulum to rotate, it is necessary to tie a cotton thread round the middle of the ball, and then pull the pendulum out of the vertical by means of the free end of the thread, which latter should then be fixed until the pendulum is quite free from vibration. The pendulum may then be started by burning the thread, and should now swing with the paper tail always in the plane of the swing, and every time it crosses the position of rest at the same time as the pendulum of the clock, the flash of light which otherwise appears in the telescope will be obscured. This will happen when the pendulums are crossing the central point in the same direction, and when they are crossing in opposite directions. Observation of

the times at which this happens constitutes the **Method of Coincidences**, which should be conducted in the following way. When a coincidence is approaching, the observer must note the clock time, then continue the counting of the seconds mentally, and be prepared for the exact coincidence to be indicated by the absence of the flash. As soon as this occurs, and the time is written down, the observation of this coincidence is finished. Let the same method be employed for finding the time of successive coincidences.

It may happen, if the pendulums have very nearly the same time of oscillation, that for several successive swings no flash appears. We must then take the mean between the two times at which obscuration commences and at which it ends. Often it will be found easier to take only the coincidences which take place when the pendulums are crossing the centre in the same direction.

The theory of the method is exceedingly simple. Let n be the number of seconds as observed by the clock be-

Fig. 94.—BORDA'S PENDULUM.

tween two successive coincidences. We must consider two cases. *First*, let the simple pendulum be going faster than the clock pendulum, then, after a coincidence, the former will gain on the latter a little at each swing; and this will continue until a whole oscillation has been gained, when they will again cross the centre simultaneously; but in the n seconds which have elapsed the simple pendulum has made $n+1$ swings, so that its time of vibration is $\frac{n}{n+1}$ seconds. *Secondly*, let the simple pendulum be going more slowly than the clock pendulum. Here in the n seconds the former will make $n-1$ swings, which gives a time of vibration of $\frac{n}{n-1}$ seconds. If coincidences only in the same direction be observed, these formulæ become respectively $\frac{n}{n+2}$ and $\frac{n}{n-2}$. The method is one which admits of extreme accuracy. For instance, if $n=251$, and the simple pendulum be the slower, then $\frac{n}{n-1}=\frac{251}{250}=1{\cdot}0040$. Supposing now an error of a second be made, and that n becomes 250, then $\frac{250}{249}=1{\cdot}004016$; the time would therefore be correct to within the extremely small amount of $\frac{16}{1{,}000{,}000}$ of a second.

Having obtained the time, the length of the pendulum and the diameter of the sphere should be ascertained by the means already described.

Example.—Let us take as an example one of the observations[1] made by MM. Biot and Matthieu for determining the length of a seconds pendulum at Dunkerque, 27th February 1809:—

I.—Observation of Coincidence.

	h.	m.	s.	m.	s.	
Coincidence did not exist at .	7	53	35			Matthieu.
" " " . . .	"	53	40			Biot.
" was exact at . . .	"	54	10	54	27·5	Biot.
" " . . .	"	54	45			Matthieu.
" passed at	"	55	0			Biot.
" "	"	55	35			Matthieu.

[1] Biot's *Astronomie Physique*, tome ii., Appendix.

Barometer, 774·25 mm. Semi-amplitude, 1° 8′ 33″.
Thermometer of barometer, 9°·1. Temperature of air, 7°·23.

Pendulum slower than the clock.

II.—OBSERVATION OF NEXT COINCIDENCE IN SAME DIRECTION.

	h.	m.	s.		
Coincidence did not exist at	9	50	30		Matthieu.
,, ,, ,,	,,	50	40		Biot.
,, was exact at	,,	51	20	m. 51 s. 25	Biot.
,, ,,	,,	51	30		Matthieu.
,, passed at	,,	52	35		Biot.
,, ,,	,,	52	45		Matthieu.

Barometer, 774·05 mm. Semi-amplitude, 0° 41′ 44″.
Thermometer of barometer, 9°·6. Temperature of air, 7°·78.

Interval between coincidences—

h. m. s. 9 51 25 − h. m. s. 7 54 27·5 = h. m. s. 1 56 57·5 = 7017·5 clock oscillations.

In this time the pendulum made 7015·5 oscillations, hence the time of oscillation of the pendulum was $\frac{7017 \cdot 5}{7015 \cdot 5}$ seconds as given by the clock,[1] which requires correction for amplitude and rate of the clock.

The length of the pendulum was obtained by the method of Lesson L., the distance between the knife-edge and the end of the measuring-rod being ascertained by a *comparateur.* In obtaining the reduced length of the pendulum corrections were made for (1) expansion by heat; (2) moment of inertia of bob of pendulum, suspending wire, and the small cap by which the end of the wire was attached to the bob; (3) air displacement.

155. *Local Values of Gravity.*—The following table gives in the C. G. S. system the value of gravity at different places, and also the length of a pendulum beating seconds. This last quantity is obtained from the formula $t = \pi\sqrt{\frac{l}{g}}$ by making $t = 1$, which will then give $l = \frac{g}{\pi^2}$.

[1] The clock actually used by the observers was a special one, whose pendulum made 100,000 + 35·265 oscillations in a mean solar day.

TABLE W.

LOCAL VALUES OF GRAVITY.

	Intensity of Gravity.	Length of Seconds Pendulum.
Equator . .	978·10	99·103
Latitude 45° . .	980·61	99·356
Paris . . .	980·94	99·390
Greenwich . .	981·17	99·413
Berlin . . .	981·25	99·422
Dublin . . .	981·32	99·429
Manchester . .	981·34	99·430
Edinburgh . .	981·54	99·451
Pole . . .	983·11	99·610

III.—APPLICATION OF ELECTRICITY TO TIME MEASUREMENTS.

156. The solution of many experimental problems which require a knowledge of time may be facilitated by electrical arrangements; some of these we therefore propose to use.

LESSON LII.—**Use of Electro-Chronograph.**

157. *Exercise.*—To make use of a Morse telegraph instrument as a chronograph.

Apparatus.—The receiving portion of an ordinary Morse telegraph instrument. The chief details of the instrument ordinarily used in England (that of Siemens and Halske) are seen in Fig. 95. An electro-magnet M is arranged to attract an armature A, which is pivoted at *p*, so that when the armature is attracted by the magnet the distant end with its *inking-wheel* W will rise. Two *friction wheels* F, driven by clockwork, cause a narrow ribbon of paper *r* to unwind from a reel and travel in the direction indicated by the arrows. The inking-wheel W also is caused to revolve by the clockwork, so that its edge is kept continually wet with ink supplied by the reservoir I, and hence whenever

W comes into contact with the paper it makes a mark. For manipulation with the Morse instrument a "key" will be required for making and breaking the circuit, also a constant battery of a suitable kind, and wires for making connections.

Time must be given by a clock beating seconds, fitted with electrical connections. For this purpose a fine wire runs down the pendulum, commencing from the lower part

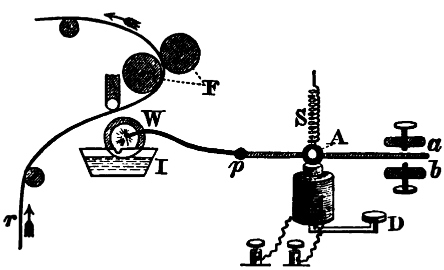


Fig. 95.

of the spring by which the pendulum is suspended and terminating at the bottom of the bob, where the fine wire is connected with a piece of platinum wire of short length. The end of the latter projects about $\frac{1}{4}$ inch beyond the end of the bob. A wooden cup with a small drop of mercury is placed so that as the pendulum swings to and fro the platinum wire may pass through the mercury at the middle of its swing. Two binding screws must be placed outside the clock case—one in connection with the framework of the clock, the other in connection with the drop of mercury.

Method.—The Morse instrument must first be adjusted, so that when it is connected with the battery and key, and when the clockwork has been set agoing by moving the switch, distinct *dots* may be produced whenever the circuit is momentarily closed. The main adjustments which re-

quire attention are the following:—(1) Adjust the stops *a* and *b*, so that the play of the armature may be small; (2) raise the electro-magnet by the screw D until only a thin streak of light is visible between the armature and the magnet; (3) adjust the tension of the spring S, so that any tendency that the magnet may have to retain its hold upon the armature, due to *residual magnetism*, may be overcome; (4) the paper must only just be touched by the inking-wheel when the armature is attracted. These pre-

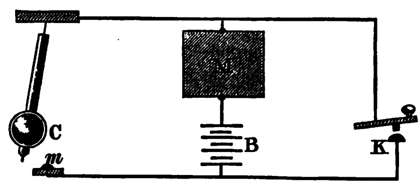


Fig. 96.

liminary adjustments having been completed, and the marking proving quite satisfactory, the student may proceed to make the connections and test the several arrangements now to be described.

I. Arrange the clock and key circuits in the manner shown in the diagram (Fig. 96), where C is the pendulum, *m* the drop of mercury, B the battery, M the Morse instrument, and K the key. The key circuit remaining open, then evidently every time the pendulum passes through the mercury the circuit will be complete, and a dot will be registered on the paper ribbon of the Morse instrument. We shall thus obtain a series of dots representing intervals of one second. If now the key be pressed momentarily, the second circuit will be closed, and an additional mark will appear on the paper (unless the key happens to be pressed at the very moment when the pendulum is passing through the mercury). If the key be pressed a second

time, a second additional mark will appear, and the interval between the two marks will indicate the lapse of time which we have wished to record. The paper will be marked after this manner—

- — - - - - — -
a *b*

which shows that the interval between the two signals was about 4·3 seconds, in which the observer's signals, generally distinguishable by their greater length, are shown at *a* and *b*. Such an arrangement constitutes a **Make Circuit Chronograph.**

II. The electro-chronograph is not free from the *loss of time* of which we have already spoken; but this will not cause any error if the length of time that elapses after pressing the key until the soft iron becomes a magnet sufficiently strong to attract the armature is the same on each occasion. It is thought that any possible error due to the variation of this quantity will be much diminished if we arrange our circuits so that the signals produced by the clock and the observer may be indicated by breaks in an otherwise continuous line, in which case the battery is always in action. Thus a record would appear as follows:—

—— —— - — —— —— —— - —
a *b*

To obtain signals of this kind the circuits must be made so that when either the clock or the key circuit is complete the current will no longer pass through the Morse, but be *short circuited* through connecting wires of comparatively low resistance, as shown in the diagram (Fig. 97), the lettering being as before. This arrangement constitutes a **Break Circuit Chronograph.**

In using the records obtained by either of these methods the time must be reckoned from the *beginning* of the recorded signal. To aid in estimating correctly the fractions of a second, a glass scale, ruled like the diagram

(Fig. 64, Chapter on Density), is convenient, but the line AB must be divided into a number of *equal* parts. To use this scale apply it to the Morse ribbon, so that the line AB may be parallel with the recorded marks which should cross the diverging lines of the scale at some place, such as CD (Fig. 64), so chosen that the line CD is equal in length to the distance between the two marks indicating seconds. This distance will be divided in the same proportion as AB,

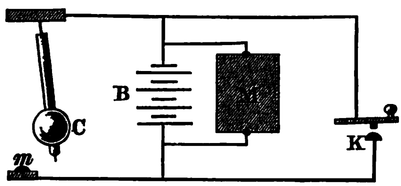


Fig. 97.

so that if AB be divided into twenty equal parts, CD will also be divided into twenty equal parts, and the position of any intermediate signal may be correctly estimated.

Since the total number of seconds is obtained by counting, and the fractional parts estimated with reference to the two adjacent marks, it will be seen that the only assumption made as to the uniformity of the motion of the paper is, that it remains constant during the period of one second. The regularity of the motion in any ordinary Morse being sufficient to admit this, a governor for maintaining uniformity of speed is thus unnecessary.

III. The Morse instrument may also be employed as a **counter**. For instance, to determine the time of vibration of a pendulum, either of the connections I. or II. may be used, the clock pendulum being replaced by the experimental pendulum. The key is smartly pressed, and the time by the clock or chronometer observed. The pendulum may then be allowed to continue swinging for a long period,

the observer making signals at stated times, as long as may be desired. It remains only then to count the number of beats that have been made in the given time.

IV. The instrument may likewise be used in comparing pendulums by the method of coincidences, the circuits being so arranged that the times at which the clock and experimental pendulums pass the central point together may be recognised on the paper ribbon.[1]

LESSON LIII.—The Tuning-Fork Electro-Chronograph.

158. *Exercise.*—To find the time taken by a body falling freely from a height.

Apparatus and Method.—A metal cylinder or drum mounted so as to be capable of rotation by clockwork or otherwise has tightly wrapped round it a band of paper having a smooth surface. The paper is coated with a thin covering of soot by holding it over a smoky paraffin flame. In conjunction with the cylinder is used a large tuning-fork of known time of vibration, having a small metallic style or pointer attached to the end of one prong. The tuning-fork is supported on a firm stand, so that when it is set in vibration and its style brought into contact with the rotating drum a sinuous line may be marked on the paper.

The body whose time of fall is to be observed should be an iron ball, which is held by an electro-magnet fixed about 9 to 12 feet from the floor. When the electro-magnetic circuit is broken, the ball falls, and just before it reaches the ground it breaks a second circuit. The instants of time at which the body commences to fall, and at which it breaks the second circuit, are recorded on the smoked cylinder by a simultaneously induced current from an induction coil which causes a spark to pass from the style

[1] For a description of this method, see *Proceedings Physical Society*, vol. iii.; Professor Ayrton and Perry on the *Value of Gravity in Japan.*

of the tuning-fork to the drum. The spark in its passage, by removing a spot of dust, gives a recognisable signal.

Fig. 98 shows the apparatus and connections. The electro-magnet E, made of very soft iron, has its lower surface separated from the ball B by a piece of paper, so that the ball will be released the more readily when the circuit is broken. The induction coil I should be of large size, and have its vibrating contact-breaker screwed up so

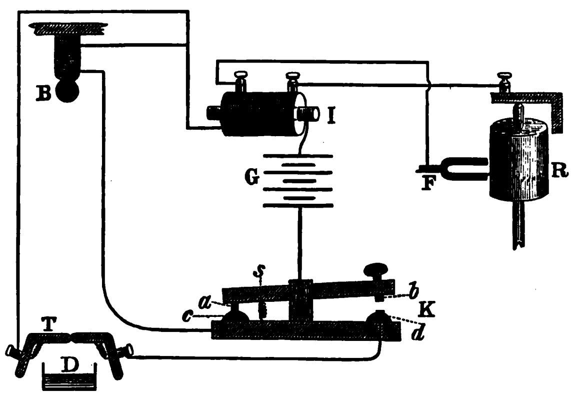


Fig. 98.

as not to be movable. At K is a "Morse key." The contact to be broken by the fall of the ball is at T; it may conveniently be made in the form of two trap-doors, having the inner edges covered with platinum foil, and connected with two binding screws. The effect of the concussion of the ball on the trap-doors may be lessened by india-rubber. After the ball has passed through T, it is received by a box of sawdust D. The rotating drum is shown at R, and the tuning-fork at F. The connections, as shown in the figure, must be made with the battery of Grove's cells G.

The normal position of the key is when the lever arm

ab rests with its end *a* against the contact *c*, being held in this position by the spring *s*. If the connections be followed, it will be seen that the current will pass through the primary wire of the induction coil on its way to the magnet, the circuit being complete as long as the key remains as figured in Fig. 98. We shall now suppose that the drum is set revolving and the tuning-fork made to vibrace, and placed with its style touching the smoked surface. As soon as the key is pressed the ball will fall, and simultaneously with the beginning of its motion—owing to the breaking of the primary circuit—a current is induced in the secondary circuit, and a spark will pass between the drum and the tuning-fork. Directly afterwards, by pressing the key still further, the trap-door circuit is made, so that when the ball in its fall breaks this second circuit a second spark passes to the drum.

Supposing the experiment to have been successful, the tracing produced by the style should be as shown in Fig. 99, a dot appearing at *a* and at *b*. The number of vibra-

Fig. 99.

tions and parts between these points must be counted, which will give us the time of fall. Suppose, for instance, that the rate of the tuning-fork is 250 single vibrations per second, and that 151·2 vibrations have been counted, then the time of fall will be $\frac{151\cdot2}{250} = \cdot6048$ second.

The height of fall should be measured, and the time compared with that given by the formula $t = \sqrt{\frac{2s}{g}}$, where s is the height and g the local value of gravity. A fairly close agreement should be obtained, inasmuch as the method is capable of giving good results.

IV.—MOMENTS OF INERTIA.

159. *Angular Velocity.*—Consider the body (Fig. 100) rotating about the centre S, and let P and Q denote the position at a given instant of two particles. The velocity of P will be greater than that of Q, for the velocity at any point depends upon its distance from the centre of rotation, the greater the distance the greater being the velocity of the point. The velocity of a rotating body as a whole cannot thus well be stated with reference to the actual velocities of its separate particles. Instead of this we take the angular amount of turning of the body in a unit of time. Thus, if in the unit of time the line SP turns through the angle PSP′, the line SQ will describe an equal angle QSQ′, and the amount of turning measured in this way must be the same for all other points. The amount of turning per unit of time is called the *angular velocity of the body.* It is measured in circular measure; thus suppose that in one second P or any other point passes through n degrees, then the angular velocity (usually denoted by ω) will be

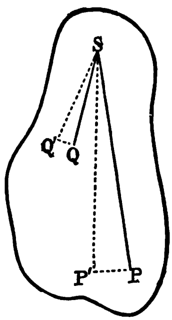


Fig. 100.

$$\omega = \frac{n\pi}{180} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since in the unit of time P has described the arc PP′, the velocity of P will be represented by the length of this arc; or, denoting by r the radius SP, the velocity of P will be

$$r\omega, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and a like expression would denote the velocity of any other point.

160. *Energy of a Rotating Body.*—The amount of energy in a body moving without rotation (the motion being

pure translation) is denoted by the expression $\frac{1}{2}MV^2$, where M is the mass and V the velocity of the body. In making use of this expression to find the energy of a rotating body we perceive that we must consider the velocities and masses of the individual particles of which it is composed. Calling $m_1, m_2 \ldots$ the masses of the particles at the points P, Q . . . and $r_1, r_2 \ldots$ their distances from the centre of rotation, and ω the common angular velocity, then the total energy of the rotating body amounts to

$$\tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \tfrac{1}{2}m_3r_3^2\omega^2 \ldots$$

or

$$\tfrac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 \ldots) \qquad (3)$$

The expression within the brackets is called the **Moment of Inertia** of the body. If M be the total mass of the body, then

$$M = m_1 + m_2 + m_3 \ldots \qquad (4)$$

and if we find a length k, such that

$$Mk^2 = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 \ldots \qquad (5)$$

the length k is called the **Radius of Gyration.**

The problem of finding the value of the expression Mk^2 with reference to a *point*, *axis*, or *plane* is in general solved by the aid of the integral calculus.

The following table gives the most useful cases, M in each case being the mass of the body, and I signifying the required moment.

TABLE X.

MOMENTS OF INERTIA.

Uniform thin Rod, axis at end, length $= l$ . . .	$I = M\frac{l^2}{3}$
,, ,, axis through middle, length $= l$	$I = M\frac{l^2}{12}$
Rectangular Lamina, axis through centre and *parallel* to one of sides, a and b length of sides, a the side bisected	$I = M\frac{a^2}{12}$

Rectangular Lamina, axis through centre and *parallel* to one of sides, a and b length of sides, b the side bisected	$I = M\frac{b^2}{12}$
,, ,, axis through centre and *perpendicular* to the plane, a and b length of sides . . .	$I = M\frac{a^2+b^2}{12}$
Rectangular Parallelopiped, axis through centre and *perpendicular* to a side; a, b, and c length of sides—	
,, ,, (1) axis perpendicular to side contained by a and b.	$I = M\frac{a^2+b^2}{12}$
,, ,, (2) axis perpendicular to side contained by a and c.	$I = M\frac{a^2+c^2}{12}$
,, ,, (3) axis perpendicular to side contained by b and c.	$I = M\frac{b^2+c^2}{12}$
Circular Plate, axis through centre perpendicular to the plate, radius $= r$ . . .	$I = M\frac{r^2}{2}$
,, ,, axis any diameter, radius $= r$ . .	$I = M\frac{r^2}{4}$
Circular Ring, axis through centre perpendicular to plane of ring, outer radius $=$ R, inner radius $= r$	$I = M\frac{R^2+r^2}{2}$
,, ,, axis any diameter, radii as before .	$I = M\frac{R^2+r^2}{4}$
Right Cylinder, axis the axis of figure, $r =$ radius of section	$I = M\frac{r^2}{2}$
,, ,, axis through centre perpendicular to axis of cylinder of length l . .	$I = M\left(\frac{l^2}{12}+\frac{r^2}{4}\right)$
Hollow Cylinder, axis the axis of figure, outer radius $=$ R, inner radius $= r$	$I = M\left(\frac{R^2+r^2}{2}\right)$
Right Cone, axis the axis of figure, $r =$ radius of base	$I = \frac{3}{10}Mr^2$
Sphere, axis any diameter, $r =$ radius	$I = \frac{2}{5}Mr^2$

161. *Moments of Inertia for Parallel Axes.*—If the

moment of inertia with respect to an axis passing through the centre of gravity be known, we can easily deduce from it the moment of inertia with respect to another axis parallel to the first. Let I be the known moment of inertia through the centre of gravity, and I′ that through the parallel axis; also let M denote the whole mass of the body, and a be the distance between the axes.

Then[1]

$$I' = I + Ma^2.$$

Example.—Find the moment of inertia of a right cylinder about any edge of the cylinder, where r = radius of section.

$$\text{Here } I = \frac{Mr^2}{2} \text{ and } Ma^2 = Mr^2 \therefore I' = \frac{Mr^2}{2} + Mr^2 = \frac{3}{2}Mr^2.$$

162. Where the body is of irregular figure, and the moment of inertia cannot be readily obtained by calculation, it is necessary to resort to the following experimental method.

Lesson LIV.—Determination of Moment of Inertia.

Exercise.—To find the moment of inertia of a magnetometer magnet.

Apparatus.—A magnetometer vibration magnet of the Kew pattern (see Vol. II.) This is a hollow tube of steel, with a lens at one end, and a glass scale at the other. The support for the magnet consists of two short hollow tubes attached one above the other, in the lower of which the magnet is placed, so that the short tube is in the middle of the magnet. The magnet is then suspended by silk fibres, so that it hangs horizontally. Its scale is viewed by a telescope. In conjunction with the magnet there is used a brass cylinder truly turned, so that its exact dimensions may be ascertained. This, when in use, is

[1] The proof of this important property will be found in many text-books; see, for instance, Williamson's *Integral Calculus* or Twisden's *Practical Mechanics*, one of which might be consulted.

placed in the second short supporting tube above the magnet. A chronometer beating half-seconds will also be required.

Method.—Things being arranged so that the scale of the magnet is distinctly seen through the telescope, the magnet is set vibrating through a small arc, and the time of vibration ascertained by the method of passages. The brass cylinder is then added, and the time of vibration again ascertained.

To obtain the dimensions of the cylinder it will be necessary to use the dividing engine. The mass of the cylinder must be found by weighing. From these data the moment of inertia of the cylinder may be obtained by the formula

$$I = M\left\{\frac{l^2}{12}+\frac{r^2}{4}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

Let us call t = time of oscillation of the magnet alone, and t' = its time with the added cylinder, and let I' be the required moment of inertia.

Each system constitutes "a magnetic pendulum with torsion" (see Art. 164). When the magnet is vibrated alone, the moment of inertia of the system being I', the time of vibration is

$$t = \pi\sqrt{\frac{I'}{\mu H + T}} \quad . \quad . \quad . \quad . \quad (2)$$

where $\mu H + T$ is the directive force due to magnetism and torsion. When the cylinder of moment of inertia I is added, the time of vibration becomes

$$t' = \pi\sqrt{\frac{I+I'}{\mu H + T}} \quad . \quad . \quad . \quad . \quad (3)$$

for the directive force remains the same.

From (2) and (3)

$$\frac{t^2}{t'^2} = \frac{I'}{I+I'} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and finally from (4)

$$I' = I\frac{t^2}{t'^2 - t^2} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Example.—The dimensions of the bar were known to be in the British system as follows:—

$M = 1009{\cdot}264$ grains, $l = {\cdot}3209$ foot, $r = {\cdot}0165$ foot.

Time of magnet alone $= t = 4{\cdot}38$ seconds.
" " with bar $= t' = 7{\cdot}75$ "

$$I' = 1009{\cdot}264\left(\frac{{\cdot}3209^2}{12} + \frac{{\cdot}0165^2}{4}\right)\left(\frac{4{\cdot}38^2}{7{\cdot}75^2 - 4{\cdot}38^2}\right)$$
$$= 4{\cdot}097 \text{ foot, grain.}$$

We shall convert this result into C. G. S. measure, making use of the fact that the moment of inertia is a quantity embracing mass × length squared,[1] which at once enables us to write

$$x \text{ gramme (centimètre)}^2 = I' \text{ grain (foot)}^2.$$

$$x = I' \frac{\text{grain}}{\text{gramme}}\left(\frac{\text{foot}}{\text{centimètre}}\right)^2.$$

$$= I' \frac{1}{15{\cdot}432}\left(\frac{1}{{\cdot}0328}\right)^2.$$

$$= I' \times 60{\cdot}24 = 246{\cdot}803 \text{ centimètre, gramme.}$$

V.—The Compound Pendulum.

163. To derive the formula for the compound pendulum let us imagine an irregularly-shaped body (Fig. 101) to be swung from a **Centre of Suspension** at S, also let G be the centre of gravity of the system, and let it oscillate in the same time as a simple pendulum of length SO. The point O is called the **Centre of Oscillation** of the

[1] This is called the *Dimensions of Moment of Inertia.* The student is advised to read *Units and Physical Constants,* by Professor Everett, Chapter I., in order to understand the true nature and utility of dimensions when changing from one set of units to another.

pendulum. The distance SO is called the **Length of the Equivalent Simple Pendulum.** Let SG be denoted by l, and SO by λ, and let M be the mass of the system.

Suppose now that a simple pendulum of mass M and length l is displaced so as to be at an angle θ with the vertical. The energy which it will have when it has been let go and falls a certain distance will be Mgh—M denoting its mass, g the force of gravity, and h the vertical height through which it has fallen.

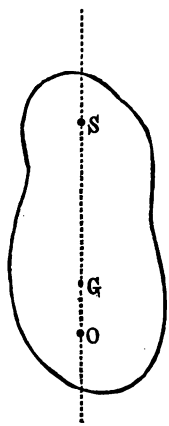


Fig. 101.

Now let the compound pendulum (Fig. 101) be displaced so that SG makes an angle θ with the vertical. The energy which it will have when it is let go and falls a distance h will be Mgh—that is to say, the same as for the simple pendulum. If ω be the angular velocity at the end of the fall, then (Art. 160) the energy of the compound pendulum at this moment will be $\frac{1}{2}M\omega^2k^2$, where k is the radius of gyration of the pendulum as swung from S.

Hence it is clear that

$$\tfrac{1}{2}M\omega^2k^2 = Mgh \quad . \quad . \quad . \quad . \quad . \quad (1)$$

but

$$h = l(\cos\theta_1 - \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where θ_1 is the angle SG makes with the vertical after G has fallen the distance h. So that

$$\omega^2 = \frac{2gl}{k^2}(\cos\theta_1 - \cos\theta) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Consider now the simple pendulum of length SO $= \lambda$. Let it be displaced through an angle θ and then allowed to fall to the angle θ_1, then the velocity v of a particle at

O at the end of this fall through the height h_1 will be

$$\begin{aligned} v^2 &= 2gh_1 \\ &= 2g\lambda(\cos\theta_1 - \cos\theta) \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

or the pendulum will have an angular velocity

$$\omega_1{}^2 = \frac{2g}{\lambda}(\cos\theta_1 - \cos\theta) \quad . \quad . \quad . \quad . \quad (5)$$

Comparing (3) and (5) we see that for all angular displacements $\omega = \omega_1$, if

$$\frac{l}{k^2} = \frac{1}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, when this relation is satisfied, the simple pendulum of length $= \lambda$ will always be moving with the same angular velocity as the compound pendulum, so that the compound pendulum will oscillate as a pendulum of length λ, and finally

$$\lambda l = k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or, in a compound pendulum, *The distance of the centre of gravity from the centre of suspension* (l) *multiplied by the distance between the centre of suspension and the centre of oscillation* (λ) *is equal to the square of the radius of gyration* (k). Multiply both sides of (7) by M, then

$$\mathrm{M}\lambda l = \mathrm{M}k^2 = \mathrm{I}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

I_1 being the moment of inertia of the system round S.

Now let I_2 be the moment of inertia round O, and I that round the centre of gravity G. Suppose likewise that the system is now suspended from O, oscillating on an axis parallel to its former axis at S. Let λ' be now the length of the equivalent simple pendulum, which represents the time of vibration of the system. It is clear that the distance from the new centre of suspension to the centre of gravity is now $\lambda - l$. Hence the equation similar to (8) will be

$$\mathrm{M}\lambda'(\lambda - l) = \mathrm{I}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now, by Article 161,

$$\mathrm{I}_1 = \mathrm{I} + \mathrm{M}l^2, \text{ and } \mathrm{I}_2 = \mathrm{I} + \mathrm{M}(\lambda - l)^2.$$

Hence

$$I_2 = I + M\lambda^2 + Ml^2 - 2M\lambda l = I_1 + M\lambda^2 - 2I_1 = M\lambda^2 - I_1 = M\lambda\,(\lambda - l).$$

Hence also from (9)

$$M\lambda'(\lambda - l) = M\lambda(\lambda - l).$$

Hence $\lambda' = \lambda$. In other words, *the centres of suspension and oscillation are reciprocal.*

Again, since for a simple pendulum of length λ

$$t = \pi\sqrt{\frac{\lambda}{g}},$$

and since, for the compound pendulum under consideration,

$$\lambda = SO = \frac{k^2}{SG} = \frac{Mk^2}{M(SG)} = \frac{I}{Ml},$$

where I is the moment of inertia, and l the distance of the centre of gravity from the point of suspension; therefore,

$$t = \pi\sqrt{\frac{I}{Mgl}} \quad . \quad . \quad . \quad . \quad (10)$$

which is the general expression for a pendulum oscillating under the action of gravity.

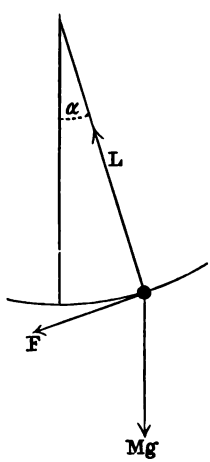


Fig. 102.

164. *Directive Force.*—Let us inquire what is the meaning of the denominator in the last expression. Consider the pendulum of length L (Fig. 102) when making an angle α with the vertical. If the bob be of mass M, the force of gravity acting upon it will be Mg, but part of this will be resisted by the string; the effective portion F, it may at once be seen, $=$ Mg sin α, whose moment with respect to the centre of suspension is MgL sin α, but if the angle α is small, sin $\alpha = \alpha$, and

therefore the moment of the effective force is $MgLa$, so that for all small angles

$$\frac{\text{Moment of Effective Force}}{a} = MgL.$$

This latter quantity is called the **Directive Force.** Let $D = MgL$; formula (10) then assumes the general form

$$t = \pi \sqrt{\frac{I}{D}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which is applicable to all pendular motions.

This formula has many important applications in the laboratory. Let us examine three cases—

(1.) *The Torsion Pendulum.*—Since the effective force varies as the angle of twist, we write simply

$$t = \pi \sqrt{\frac{I}{T}} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where T is the moment of torsion for unit twist.

(2.) *The Magnetic Pendulum.*—Here a magnet is supported horizontally by a torsionless thread and set in vibration. If the magnet be of moment μ, and H is the horizontal component of the earth's magnetism, the directive force will be μH, and the formula becomes

$$t = \pi \sqrt{\frac{I}{\mu H}} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

(3.) *The Magnetic Pendulum with Torsion.*—If the thread or wire supporting the magnet have sufficient torsion to affect the rate of vibration, the moment of the torsion must be taken into account, and the formula becomes

$$t = \pi \sqrt{\frac{I}{\mu H + T}} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

VI.—GRAVITATION (*Resumed*).

165. *Captain Kater's Pendulum.*—In the early attempts at measuring the intensity of gravitation the main hindrance to accuracy was the difficulty in obtaining the exact length of the pendulum. If instead of a simple pendulum with a flexible wire we use a rigid compound pendulum, and find the length of its equivalent simple pendulum by vibrating the body first from one point and then from another until the times of vibration are the same, then since the distance between these two fixed points has only to be obtained in order to furnish the equivalent length of the simple pendulum, there will no longer be any difficulty in ascertaining the exact length. This was the principle employed by Captain Henry Kater, and described by him in the *Philosophical Transactions* for 1818, entitled, "An account of experiments for determining the length of the pendulum vibrating seconds in the latitude of London."

LESSON LV.—**Determination of Gravity by Kater's Method.**

Exercise.—To adjust a Kater's pendulum and find the length of the equivalent simple pendulum, thence deducing the value of gravity.

Apparatus.—A model of Kater's pendulum. This consists of a long brass rod $1\frac{1}{2}$ inch wide and $\frac{1}{8}$ inch thick; it has two fixed knife-edges k and k' (Fig. 103) near its ends, the distance apart being about 39·4 inches. Near one of the knife-edges is a bob B of 2 lbs. 7 oz. in weight and $3\frac{1}{2}$ inches diameter. A weight A of $7\frac{1}{2}$ oz. is placed about 5 inches from the knife-edge k, this weight being capable of sliding up and down the rod and being fixed in any position by a clamp screw. A smaller weight or slider S of 4 oz. is placed near the middle of the bar; this has a screw for

giving it a small motion up and down the bar. Attached to each end of the rod is a light rod of wood, d and d', painted black, to act as tailpieces in determination of the time of oscillation, terminated by two pointers, p and p', for observation of the amplitude of vibration determined by the extent of swing over a graduated scale.

For measuring the length of the bar two reading microscopes and a standard measure will be required, and for obtaining the time of oscillation the seconds clock and other apparatus used in the method of coincidences (Lesson LI.).

Method I.—Place the pendulum in front of the clock and

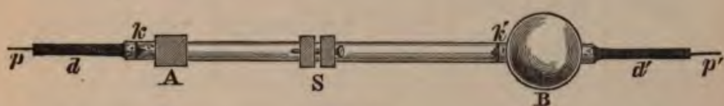


Fig. 103.

let it oscillate from the knife-edge k', and determine the time by the method of the coincidences; then invert the pendulum and oscillate it from k. If the pendulum in the first case has a time of oscillation less than in the second case the weight A must be moved nearer its knife-edge, the process being repeated until the times about both knife-edges are nearly the same, the final adjustments being made by the slider, the weight being always moved towards that knife-edge about which the time of oscillation is the greater.

To determine the length of the pendulum, it must be laid horizontally in a wooden frame. Two microscopes, each with a micrometer scale, are then adjusted until the knife-edges are seen in focus. The pendulum is then removed and a scale substituted for it, the microscopes remaining meanwhile undisturbed. The scale, by raising or lowering, is brought into good focus, and the length is then directly read off.

Example.—We shall take for an example an observation

made by Kater, which will show the method he adopted of recording his results.

19*th June* 1818.

SLIDER 18 DIVISIONS, BAROMETER 29·7 INCHES, CLOCK LOSING 0·33 SECONDS, GREAT WEIGHT ABOVE.

Temp.	Time of coincidence.	Arc of vibration.	Mean arc.	Interval in seconds.	Vibrations in 24 hours.	Correction for arc seconds.	Vibrations in 24 hours.
	m. s.						
66°·8 F.	29 12	1°·29					
			1°·18	518	86066·40	2·28	86068·68
66°·8	37 50	1°·07					
			1°·00	520	86067·70	1·63	86069·33
	46 30	0°·93					

Mean . . . . 86069·00
Clock correction . −0·33

Corrected number of vibrations in 24 hours . 86068·67

The great weight was placed below, and the time found to be . . . . 86064·54

The movable weight was then placed nearer its knife-edge and the observation repeated.

Method II.—Instead of adopting the tedious process of altering the adjustments until the time of oscillation from either of the two knife-edges is the same, we may, when the times are approximately equal, use a formula obtained as follows. Let k be the radius of gyration of the pendulum with reference to an axis passing through its centre of gravity. Let h and h' be the distances of the centre of gravity from the knife-edges, about which the times of vibration are respectively t and t'. Calling M the mass of the pendulum, we shall have

$$t=\pi\sqrt{\frac{\mathrm{M}(k^2+h^2)}{\mathrm{M}hg}} \quad . \quad . \quad . \quad . \quad (1)$$

and

$$t' = \pi \sqrt{\frac{M(k^2 + h'^2)}{Mh'g}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which give respectively

$$hgt^2 = \pi^2(k^2 + h^2) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$h'gt'^2 = \pi^2(k^2 + h'^2) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating k from (3) and (4) we obtain

$$g(ht^2 - h't'^2) = \pi^2(h^2 - h'^2) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or

$$\frac{\pi^2}{g} = \frac{ht^2 - h't'^2}{h^2 - h'^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (6) may at once be thrown in the form [1]

$$\frac{\pi^2}{g} = \frac{t^2 + t'^2}{2(h + h')} + \frac{t^2 - t'^2}{2(h - h')} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The first term of this expression can be determined completely, for $h + h'$ is the distance between the two knife-edges; the second term is very small, for t, it is supposed, differs little from t', hence $t^2 - t'^2$ is very small, while in a Kater pendulum $h - h'$ is relatively large, so that if we determine experimentally or by calculation the *approximate* distances of the centre of gravity from each knife-edge equation (7) will enable us to determine the value of g with accuracy.

166. *Bessel's Method.*—The difficulty of exactly measuring the length of the pendulum has also been surmounted by Bessel in an ingenious manner. A pendulum was suspended first from a fixed point at a distance l above the centre of the sphere, and the time t accurately determined; the wire supporting the pendulum was then shortened, and the pendulum suspended from a second fixed point of support below the first one. The bob of the pendulum occu-

[1] Routh's *Rigid Dynamics*, new edition.

pied the same position as before, the position being secured by a *lever of contact.* Let now the support be at a distance l' above the centre of the sphere, the time of vibration being t'. The value of gravity will then be given by the formula

$$g=\frac{\pi^2(l-l')}{t^2-t'^2}\left(1-\frac{2r^2}{5ll'}\right),$$

where r is the radius of the bob.

In this formula only the difference between the lengths of the pendulums occurs, except in the small factor $1-\frac{2r^2}{5ll'}$. Now, although the exact lengths of l and l' are difficult to obtain, the method adopted by Bessel allowed the determination of the difference between the lengths with great accuracy, so that the method is an exceedingly good one.

VII.—Time Measurement (*Resumed*).

167. *Estimation of Tenths of a Second.*—The eye can far more readily be taught to subdivide a small space accurately than the ear can be taught to subdivide correctly a small interval of time. Hence, whenever possible, the method is arranged so that the eye can lend assistance to the ear. Consider, for instance, a moving object passing across the field of a telescope, which we shall, for the sake of simplicity, suppose to be provided with a scale in its eyepiece. When the object is approaching the centre of the field of the telescope let the observer notice the clock time, and continue mentally counting the seconds; suppose that at the 14th second the object is one division to the left of the centre of the field of the telescope, and at the 15th second it is four divisions to the right of this centre. In the one second the object will have moved five divisions, so that the transit across the middle will have taken place at 14·2 seconds. This method, which is employed in astronomical observatories, is known as the **eye and ear**

method. The student will have an opportunity of practising it when he uses the magnetometer.

168. *Personal Equation in Time Estimations.*—Practised observers who use the eye and ear method are found to make errors in their estimations of an amount which is, under the same conditions of health, fairly constant for the same observer. This error is called the **absolute personal equation** for the eye and ear method. It usually amounts to about +·2 second—that is, an observer with this personal equation will be about ·2 second behind time. In some few cases the observer, owing to an unconscious desire to anticipate the event, may be too soon in his estimation, and establish a fixed erroneous habit of this kind.

When the electro-chronograph is used we dispense with the ear and depend upon the eye and hand. It will therefore be of importance to ascertain what effect the personal equation has upon time estimations made in this way.

Lesson LVI.—Personal Equation.

Exercise.—To find the personal equation for the eye and hand when an event is not expected, and when it is expected, and to ascertain the accuracy of the electro-chronograph method.

Apparatus.—The Morse instrument fitted as a make circuit chronograph. The clock must also be provided with the mirror, etc., for giving the flashes of light in the telescope as described in Lesson LI. These flashes may be caused to be at irregular times by an assistant revolving a shutter between the clock and telescope.

Method.—When each flash appears let the key be pressed until about twenty records have been obtained on the paper ribbon. Since the flashes in the telescope have been almost simultaneous with the pendulum passing through the mercury, and hence with the marks indicating seconds,

the time interval given on the paper between a second mark and the observer's mark will give the absolute personal error. When the flashes appear regularly, and are made to cross the field of the telescope, it will be found that at first there is a tendency to press the key before the light is at the centre of the field, but usually after a little practice this error may be overcome. Let the signals be made at alternate flashes, and continued until a number of normal records have been obtained, then measure the differences between the signals of the clock and observer, and compare with the time scale.

Example I.—Flash unexpected:—

Personal error + ·17, ·19, ·20, ·22, ·20, ·22, ·18, ·17, ·19.

II.—Flash expected; alternate flashes used:—

Personal error + ·13, ·13, ·11, ·10, ·15, ·13, ·17, ·13, ·13, ·13, ·125, ·13, ·13, ·15, ·17, ·15, ·13.

III.—The difference between the successive personal errors gives the error that would be made in signalling the time between two events. The greatest difference shown by the above figures is ·05 seconds, which represents, therefore, the greatest error due to variation of the observer's loss of time.

APPENDIX.

A.

ON THE SELECTION, CONDUCT, AND DISCUSSION OF OPERATIONS SUITABLE FOR THE PHYSICAL LABORATORY.

Selection of Operations.

1. THE statement of a physical relation may be expressed either *qualitatively* or *quantitatively*. Thus, for instance, it is of importance to know that the freezing point of water is lowered by increasing the pressure, while that of paraffin is raised by the same means, even although we may not be able to assign the exact thermometric difference that one additional atmosphere will produce in either case. Or again, it is of importance to know that between 0° and 4° C. the application of intense pressure to diminish the volume of a mass of water is accompanied by a diminution rather than by an increase of temperature, although this change of temperature may be so small that we may be unable to find its exact relation to the pressure which produces it. In these two cases our information is only of a qualitative nature.

It is desirable that the laboratory student, already familiar with the laws of energy in a theoretical manner, should perform important experiments such as those we have now indicated, if the experimental difficulties are not too great. The result will be to impress upon him the reality of these laws in a very forcible manner.

2. After having thus verified the conclusions of others, he may ultimately succeed in finding out something on his own account. He may say to himself, "If such and such laws are true, they ought, in my opinion, to lead to such and such results." He may then make the appropriate experiment and obtain the anticipated result; and although this may at first be only of a qualitative nature, yet it may be of great importance, and capable ultimately of expression in exact quantitative terms.

3. Generally, however, qualitative results are more suited for the lecture-room than for the physical laboratory;[1] thus, to show the expansion of bodies through increase of temperature is a well-known class experiment, but it would be a waste of time merely to verify the fact of expansion in the physical laboratory without at the same time attempting to measure its amount. Again, there are certain very simple and fundamental physical laws[2] the mere proof of which is below the powers of the ordinary physical laboratory, and the exceptions to which are above these powers.

The law of Boyle, for instance, tells us that as long as we keep unchanged the temperature, the chemical nature, and the mass of a gas, then the volume which it occupies will be inversely proportional to the pressure under which it is retained. Now, although this is a law of quantitative expression, it may be proved approximately in the simplest manner, the proof forming a well-known class experiment unsuited to the physical laboratory in the sense of being below its powers. On the other hand, to determine the boundary at which this law begins to fail, and the conditions of failure for gases such as oxygen, hydrogen, and nitrogen, are problems much above the powers of the ordinary laboratory.

[1] It is not intended to include in this category important experiments such as cannot readily be performed in the class-room. For example, a student may learn much about the laws of capillarity, or the phenomena of polarisation of light, without making measurements.

[2] Experiments on *the laws of motion* may be excepted in the case of junior students, for such experiments tend to give a grasp of important principles.

4. In like manner, to investigate the chemical nature of compounds on the basis of a received table of equivalents is an operation which is every day performed in a chemical laboratory; while a redetermination of the exact equivalent of an element forms one of the most difficult and perplexing problems in the whole of chemistry, and demands a special and elaborate investigation.

5. But while certain processes are above and others below the powers of an ordinary laboratory, there is a large multitude of things which the student ought to know and work at, as well as a considerable number which, when advanced enough, he may undertake with the well-grounded hope of extending our physical knowledge.

He ought first to know experimentally the instruments most frequently used in physical research, as well as the proper method of using them. And after that, when more advanced, he may employ these instruments or invent others with the view of increasing our knowledge, more especially in the direction of completing tables of physical constants.

6. Let us now suppose that a student is anxious to undertake some special physical problem which he knows has not yet been solved, and that he has in view some particular form of instrument with which to work.

He ought, in the first place, by study and by preliminary trials, to become theoretically and practically conversant with the difficulties and sources of error which will beset him in his investigation. He ought also to estimate or weigh these as in a sort of mental balance, in order to see which are the most important, and to what extent he can succeed in overcoming them, or whether any of them will present insuperable obstacles in the way of his research.

If the difficulties may be overcome, he will, let us imagine, commence operations, so that we have now to do with the conduct of these.

Conduct of Operations.

7. Under this head it may be desirable to mention some of

the most prevalent tendencies which ought to be guarded against.

In the first place there is that want of scientific perspective which induces the experimenter to display extraordinary care in discussing and overcoming some special kind of error, while those of a much larger size are left unguarded. This may sometimes arise from the circumstance that the experimenter is fond of mathematical symbols, and the point in question lends itself to an exhibition of these. Or again, he may take a pleasure in inventing the instrumental means of getting rid of this particular error. Whatever be the cause, any such one-sided treatment of an experiment is sure to operate prejudicially.

In the second place, when the process is a compound one, the second part being built upon the first, the experimenter ought not to begin the second until he has ascertained beyond doubt that the first has been securely completed. Otherwise his work will be like that of the builder who builds a strong structure on a weak foundation. The strength of his process is, in fine, the strength of its weakest part.

Thirdly, there is an error to which observations such as those of a trigonometrical or magnetical survey are peculiarly exposed. Here the observer may give great attention to all the conditions required to give him minute accuracy, and yet, from a slovenly and inaccurate reading of the *larger counts* of his instrument, the tabulated results may be very far wrong, perhaps differing from the truth by 1° or by 10°. It is possible that he may have the means of rectifying this mistake, but it is conceivable that he may not, in which case he will be in the painful and ridiculous position of having lost all his labour,—in fine, of having been in a scientific sense *penny wise and pound foolish.* And in any case his observations will be to some extent discredited, from the fact that he has allowed himself to make such gross blunders.

Fourthly, there is a tendency to commit a somewhat similar error in experiments. Let us imagine that the experimenter has obtained some very good and concordant results, having made his experiments with great care. Now, however, he relaxes his attention, becomes careless, and the result is a bad experiment.

What is he then to do, for on what principle of scientific justice can he reject this result?

And yet if he retain it he knows that it will impair the accuracy of his determination. No doubt a criterion has been devised by which he may perhaps be able to dismiss the defective experiment without scruples of conscience; nevertheless such a proceeding is extremely unfortunate, and should by all means be avoided.

Having thus detailed some of the most frequent causes of error, let us conclude this paragraph by stating that the experimenter should vary his experiments as much as he conveniently can. If, for instance, he obtains a consistent series of results with one instrument, and a consistent series of the same results with a very different instrument, and if the values given by the two instruments agree very well together, he may then have great confidence in the accuracy of his determination.

Discussion of Results.

8. *Numerical Calculation.*—After the observations have been made certain numerical calculations have generally to be employed before we obtain the final result of which we are in search. This process is termed *reduction.*

In reducing our observations we must endeavour to render this numerical process as little laborious as possible, consistently of course with due accuracy in our final results. It will be found of great importance before commencing this process to put things into a systematised and tabular form, and when applying the process to make use of tables of logarithms as much as we can.

Here we must avoid any tendency to over refinement, and refrain from using too many decimal places. Generally speaking, we ought, in our calculations, to go one place beyond that which denotes the amount of accuracy we are striving after in our result. Thus, if we wish our result to be correct to the hundredth of the whole, we ought in our calculations to be right to the thousandth of the whole, and so on. For most purposes it will be convenient to use four-figure logarithms.

9. *Approximate Formulæ.*—Very frequently in our reductions we shall have to employ mathematical formulæ, into which there enter certain small quantities and the powers of these. Here we must remember that the square of a quantity already small is as much smaller than the quantity itself as this is smaller than unity. Thus $\frac{1}{1000}$ or ·001 is already a small quantity denoting the one-thousandth part of unity, but its square, $\frac{1}{1000^2}$ or $(\cdot 001)^2$, denotes the thousandth of the thousandth, or the one-millionth of unity. And in like manner the cube of a small quantity is as much smaller than the square as the square is than the quantity, and so on.

If therefore we have a term $(1+\delta)^n$, δ being a small quantity, it will be sufficient to express it as $1+n\delta$. The following approximate expressions for formulæ containing small quantities will be found useful.[1] (Where the sign $\pm$ or $\mp$ is placed before a quantity, either the upper or the lower sign must be used all through.)

$$(1+\delta)^m = 1+m\delta \qquad (1-\delta)^m = 1-m\delta \quad . \quad . \quad (1)$$

Hence in different cases

$$\sqrt{1+\delta} = 1+\tfrac{1}{2}\delta \qquad \sqrt{1-\delta} = 1-\tfrac{1}{2}\delta \quad . \quad . \quad . \quad (2)$$

$$\frac{1}{1+\delta} = 1-\delta \qquad \frac{1}{1-\delta} = 1+\delta \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{1}{\sqrt{1+\delta}} = 1-\tfrac{1}{2}\delta \qquad \frac{1}{\sqrt{1-\delta}} = 1+\tfrac{1}{2}\delta \quad . \quad . \quad . \quad (4)$$

Also

$$(1\pm\delta)(1\pm\epsilon)(1\pm\zeta)\ldots = 1\pm\delta\pm\epsilon\pm\zeta\ldots \quad . \quad . \quad (5)$$

$$\frac{(1\pm\delta)(1\pm\zeta)\ \ldots}{(1\pm\epsilon)(1\pm n)\ \ldots} = 1\pm\delta\pm\zeta\mp\epsilon\mp n\ldots \quad . \quad (6)$$

Also if p_1 and p_2 denote two quantities only slightly different from each other, so that $p_2 = p_1+\delta$, then their geometrical mean

$$= \sqrt{p_1 p_2} = \sqrt{p_1}\sqrt{p_1+\delta} = \sqrt{p_1}\left(\sqrt{p_1}+\tfrac{1}{2}\frac{\delta}{\sqrt{p_1}}\right) = \frac{p_1}{2}+\frac{p_1+\delta}{2} = \frac{p_1+p_2}{2}$$

---

[1] Taken from Kohlrausch, *Physical Measurements.*

—that is to say, the geometrical is sensibly the same as the arithmetical mean. (For an illustration of this we may refer to Chapter III. Lesson XXIV.)

10. *Errors of Ultimate Result.*—We shall suppose that a series of observations taken with very great care, and all equally trustworthy as far as we know, have been reduced. The results of the reduction will generally appear in one of the three following forms:—

(1.) As a number of separate and apparently equally trustworthy determinations of the same thing. For instance, we may make a number of separate determinations of the density or of the volume of a body at a given temperature. These determinations of the same thing will be all very near one another in value, but yet no two of them will in all probability be exactly alike; and the question is how to use them so as to get the truth out of them, or what is most likely to be the truth.

(2.) We may have a number of equations,—let us say simple equations,—from which we are to determine two or more unknown quantities, there being more equations than unknown quantities. Here the question is how to treat a large number of equations so as to obtain from them the best possible values of these unknown quantities. Suppose, for instance, that we have a number of points of which the co-ordinates are given. Had the determination of these been perfectly accurate we have reason to believe that a straight line would have passed through all the points. But this will not now take place owing to errors in the determinations of the various co-ordinates. The question is how to utilise all the observations so as to find the most probable equation to the required line.

(3.) We may obtain simultaneous values under varying conditions of two physical quantities connected together by some unknown formula. For instance, suppose we obtain values of the maximum pressure of aqueous

vapour corresponding to various temperatures,—let us say 0°, 8°·5, 17°·2, 22°·3, 28°·4, 35°·4,—and that our object is to utilise these observations so as to construct a table giving the most probable values of the maximum pressure of aqueous vapour for all whole degrees of temperature from 0° to 40°. The question is, How can we best obtain such a result from our observations? Here, while we know that the maximum pressure of aqueous vapour is a function of the temperature, we are yet ignorant how to express this function, so that the very form of the function is here unknown, as well as the most probable values of the constants that enter into it.

We shall now discuss the method of treating these three forms of experimental results.

11. *Treatment of a Series of Determinations of the Same Thing.*—It is here supposed that all the various observations of the series have been made with very great care, and that the instrument with which they were made is accurate as far as we know, or, if not, that its error is known, and that a correction on this account has already been applied to the various observations. In this case the arithmetical mean of all the observations, obtained by adding them all together and dividing by their number, will give us the most probable value of the required quantity.

It will here be desirable to say a few words regarding the numerical estimate of probabilities.

The *probability* of a certain event or combination occurring is measured by the proportion which the number of such events or combinations bears to the number expressing all possible events or combinations. A few illustrations will make our meaning clear. Suppose we have a pack of cards, and that we are asked to draw a single card out of the pack. It is required to find the probability that this shall be the ace of hearts. Now there are fifty-two cards in all, and it is assumed that there is no reason why we should draw one of them in preference to the other; therefore, the probability that we draw the ace of hearts will be

$\frac{1}{52}$, in which the numerator denotes the selected event and the denominator all the possible events. In like manner the probability that we draw an ace will be $\frac{4}{52}$ and the probability that we draw a court card $\frac{12}{52}$.

Again, assuming it as certain that the barometer will either rise or fall to-morrow, the one result being as likely as the other, if we hazard a chance prediction and say it will rise, then the probability of such a prediction turning out correct will be $\frac{1}{2}$, the numerator representing the event which we predict, and the denominator the number of possible events which may take place, which in this case is only two. It is, in other words, an even chance whether we are right or wrong.

Now, when we have taken the mean of our series of observations we cannot be certain that this mean represents the exact truth: indeed we may be certain that it is not absolutely correct. Under these circumstances all that we can do is to assign its probable error, or, in other words, to find a value such that it is an even chance whether the error of our determination will be greater than this value, or whether it will be less. We shall not here enter into the method by which this probable error is determined by means of the theory of probabilities, but will confine ourselves to giving the result obtained by the theory. Suppose, therefore, that n denotes the whole number of observations, and that

$$\delta_1, \delta_2, \delta_3 \ldots \delta_n$$

are the differences of the individual observations from the arithmetical mean of the whole. Also let S denote the sum of the squares of the errors—that is to say,

$$S = \delta_1^2 + \delta_2^2 + \delta_3^2 \ldots + \delta_n^2$$

—then the probable error of the mean of the whole will be

$$\pm 0{\cdot}6745 \sqrt{\frac{S}{n(n-1)}};$$

while the probable error of a single observation will be

$$\pm 0{\cdot}6745 \sqrt{\frac{S}{n-1}}.$$

Let us illustrate these formulæ numerically with reference to some well-known set of observations. For this purpose we shall select the example given by Encke in his paper on Probabilities. Experiments were made by Benzenberg in the Schlebuscher coal-mines on the fall of bodies. The height through which they fell was 262 Parisian feet, and in the following table easterly deviations from the perpendicular are denoted by *plus* and westerly deviation by *minus*, the unit being the Parisian line.

The experiments were devised with the object of giving an independent proof of the rotation of the earth. Now, from astronomical theory, there ought to have been a deviation towards the east = 4·6 lines.

TABLE EXHIBITING THE RESULT OF BENZENBERG'S EXPERIMENTS.

No. of Experiment.	Deviation.	No. of Experiment.	Deviation.	No. of Experiment.	Deviation.
1	− 3·0	11	+12·0	21	+ 6·0
2	+12·0	12	+ 7·0	22	− 2·0
3	+ 3·0	13	+13·5	23	+11·0
4	+13·0	14	+11·0	24	− 4·0
5	+20·0	15	+ 9·0	25	− 9·0
6	− 2·0	16	− 8·0	26	−10·0
7	+11·5	17	+ 8·0	27	+ 8·5
8	− 4·0	18	+10·0	28	+10·0
9	+ 2·0	19	+ 7·0	29	+ 5·5
10	+ 2·0	20	+ 7·5	...	...

The mean of all these is + 5·086, which therefore denotes the most probable deviation as far as these experiments are concerned. In the following table the various departures from the mean are arranged according to their magnitudes :—

No. of Experiment.	Departure from mean.	No. of Experiment.	Departure from mean.	No. of Experiment.	Departure from mean.
29	−0·414	15	−3·914	4	− 7·914
21	−0·914	18	−4·914	1	− 8·086
12	−1·914	28	−4·914	13	− 8·414
19	−1·914	14	−5·914	8	+ 9·086
3	+2·086	23	−5·914	24	+ 9·086
20	−2·414	7	−6·414	16	+13·086
17	−2·914	2	−6·914	25	+14·086
9	+3·086	11	−6·914	5	+14·914
10	+3·086	6	+7·086	26	+15·086
27	−3·414	22	+7·086	...	...

The sum of the squares of the errors will be found to be $=1612{\cdot}0$, and hence we obtain—

Probable error of a mean of the whole $= \pm 0{\cdot}6745\sqrt{\dfrac{1612{\cdot}0}{29\times 28}} = \pm 0{\cdot}950.$

,, ,, single experiment $= \pm 0{\cdot}6745\sqrt{\dfrac{1612{\cdot}0}{28}} = \pm 5{\cdot}118.$

From the first of these two formulæ we see that it is an equal chance that the true deviation lies between $+5{\cdot}086-0{\cdot}950$ and $+5{\cdot}086+0{\cdot}950$—that is to say, between 4·136 and 6·036. As a matter of fact we know that the true deviation is 4·6, which is within these limits.

From the second of these two formulæ we argue that half the errors are likely to be greater than 5·118, while half are likely to be less; and from the table we find that sixteen errors are above this limit and thirteen below it, which is a sufficiently good result. If there were no easterly deviation there would be in the mean an error of 5·086, which is more than five times its probable error, so that the existence of an easterly deviation is rendered nearly certain by these experiments.

12. It will be desirable at this stage to make three remarks. The *first* has reference to the method of recording our observa-

tions. Suppose, for instance, that we have made a series of good observations of the height of the barometer at a particular place and time. These, let us suppose, are recorded in inches and decimals of an inch to three places. It will probably be found that the second place of decimals is the same in all, but it will not do to strike off the third place of figures and treat the remainder by the method of probabilities. For since we have supposed that they all agree in the second place, they would be identical, and their probable error would come out as zero.

This, however, would not mean that they were absolutely correct, but only correct for the second place of decimals. Had we kept in the third place and submitted them to the above process we should have found that each observation had its definite error, and that the probable error of the series was not zero.

In the *next place* it is often convenient to represent what is termed the percentage of error. For instance, in the above example, the *true* easterly deviation was 4·6, while the mean of the recorded observations gave 5·086. Now $5{\cdot}086 - 4{\cdot}6 = 0{\cdot}486$, and $\frac{0{\cdot}486}{4{\cdot}6}$ gives 10·6 per cent; 10·6 is therefore the percentage of error in the series of observations, assuming that the theoretical result 4·6 is absolutely correct. The percentage of error is made use of in Lesson XXVI.

Thirdly, in a long series of observations it will for many purposes be unnecessary to resort to the system of squares, which would, under these circumstances, entail a vast amount of labour. It will be sufficient to sum up without respect of sign the various departures from the mean supposed to represent the truth, and to divide this sum by the whole number of observations; in other words, to obtain the mean departure from the mean.

This method is especially convenient when we are engaged in the process of improving some instrument, during which we make frequent series of observations, all of which, let us suppose, are practically of the same length, with the view of testing the progress we have made. It is quite clear that as long as we continue perceptibly to diminish the mean departure from the mean in various sets of observations practically of the same length, we are making satisfactory progress with our instrument.

13. It remains to discuss the question of *weight* connected with physical observations. Now, by the weight of a given value, we understand the number of equally good observations of a determinate kind (of unit exactness) which are required to furnish by their arithmetical mean a determination equal in exactness to the given value.[1]

In order to illustrate this definition, suppose that we have a series of sixteen equally good observations, then, adopting our previous terminology, we we shall have—

$$\text{Probable error of mean of whole} = \pm 0{\cdot}6745 \times \tfrac{1}{4}\sqrt{\frac{\text{S}}{15}}.$$

$$\text{,,}\quad \text{,,}\quad \text{single observation} = \pm 0{\cdot}6745\sqrt{\frac{\text{S}}{15}}.$$

Thus we see that the probable error is inversely proportional to the square root of the number of observations. But by the above definition the weight of the mean derived from the whole series will be 16, while that derived from a single observation will be 1. The weight is therefore inversely proportional to the square of the probable error.

Suppose next that we were to split up our whole series into two parts, one consisting of six, and the other of ten observations, and then take the arithmetical mean of each part. How are we to combine these two means in order to get the most probable value? Are we entitled to add them together and divide by 2? Unquestionably not, because then we should get a final result different from that obtained by taking the mean of the whole series, which we know to be the most probable result as far as these sixteen observations are concerned.

We must therefore treat them in such a manner as to give a result identical with that of the whole series. This leads us to the following formula—

$$\text{Most probable value} = \frac{(\text{mean of } 10) \times 10 + (\text{mean of } 6) \times 6}{16}$$

or, generally, if w_1, w_2, w_3, etc., represent the weights of various

[1] This definition is borrowed from Encke.

series of observations, m_1, m_2, m_3, etc., being the means of these,

$$\text{Most probable value} = \frac{w_1m_1 + w_2m_2 + w_3m_3 + \text{etc.}}{w_1 + w_2 + w_3 + \text{etc.}}.$$

Suppose next that we have a number of sets of determinations of the same thing, made perhaps by different observers, and possibly, too, by different instruments,—at any rate, each possessing different weights,—how are we to combine these in order to get the best final results? If we know the weights of the various determinations, the formula just given will enable us to deduce the most probable combined value. But if we only know the probable errors of the various determinations, we must give each set a weight inversely proportional to the square of its probable error, and having thus ascertained the weights, then apply the formula already given.

We shall illustrate this by an example borrowed from Merriman (*Method of Least Squares*).

Example.—An angle is measured four times with a theodolite, six times with a transit, and five times with a sextant, giving the observations—

By the Theodolite.	By the Transit.	By the Sextant.
6° 17′ 5″	6° 17′	6° 17′ 20″
6 17 10	6 16	6 17 0
6 17 0	6 15	6 17 40
6 17 5	6 19	6 16 50
...	6 17	6 17 10
...	6 18	...

What are the relative weights of the means and the most probable value of the angle?

Here we have in the first place—

Mean	By theodolite	= 6° 17′ 5″	also
	By transit	= 6 17 0	
	By sextant	= 6 17 12	
Probable error of mean, derived from the formula of § 11 of this Appendix.	By theodolite	= 1″·4	
	By transit	= 23″·3	
	By sextant	= 5″·8	

Now, the relative weights being inversely proportional to the squares of the probable errors, we have—

$$\frac{\text{Theodolite}}{\text{Weight}} : \frac{\text{Transit}}{\text{Weight}} : \frac{\text{Sextant}}{\text{Weight}} = \frac{1}{1 \cdot 96} : \frac{1}{542 \cdot 9} : \frac{1}{33 \cdot 64} = 277 : 1 : 16 \text{ nearly.}$$

$$\text{Hence most probable value} = \frac{(6^\circ\ 17'\ 5'')\ 277 + 6^\circ\ 17'\ 0'' + (6^\circ\ 17'\ 12'')\ 16}{294}$$

$$= 6^\circ\ 17'\ 5'' \cdot 36$$

In the next place, with regard to the probable error of this value, we cannot do better than make use of the fact that the probable error is inversely proportional to the square root of the weight, which gives us the following proportion—

$$\text{Probable error of final value} : \text{probable error of theodolite value} :: \frac{1}{\sqrt{294}} : \frac{1}{\sqrt{277}}$$

$$\text{Hence probable error of final value} = 1'' \cdot 4 \sqrt{\frac{277}{294}} = 1'' \cdot 36.$$

14. *Treatment of a number of Simple Equations involving two or more Unknown Quantities.*—Suppose we restrict ourselves to three unknown quantities, and that we have as the result of our experiments or observations the following system of equations :—

$$a_1x + b_1y + c_1z = m_1.$$
$$a_2x + b_2y + c_2z = m_2,$$
$$\text{etc. etc.}$$

Now, if all our experiments were absolutely correct, all of the above equations would be simultaneously satisfied by the true values of x, y, z. But since this is not the case, we may express the real condition of things by the following equations :—

$$a_1x + b_1y + c_1z - m_1 = \delta_1,$$
$$a_2x + b_2y + c_2z - m_2 = \delta_2,$$
$$\text{etc. etc.}$$

in which δ_1, δ_2, δ_3 represent small differences due to unavoidable errors of observation.

Now, by the principles of the theory of probabilities, it can be shown that the most probable values of the unknown quantities are those which make the sum of the squares of the errors a minimum, and hence the method of deducing these values

may be appropriately termed the *method of least squares.* It can also be shown by the same theory that the following method of treatment will produce this result. Let us first of all multiply each equation by the coefficient of x in that equation, and we obtain

$$a_1^2x + a_1b_1y + a_1c_1z = a_1m_1,$$
$$a_2^2x + a_2b_2y + a_2c_2z = a_2m_2,$$
$$\text{etc. etc.}$$

Adding all these together we obtain one final equation, which may be expressed as follows :—

$$x\Sigma(a^2) + y\Sigma(ab) + z\Sigma(ac) = \Sigma(am) \quad . \quad . \quad . \quad (1)$$

Performing the same operation upon the coefficients of y and z we obtain in like manner

$$x\Sigma(ab) + y\Sigma(b^2) + z\Sigma(bc) = \Sigma(bm) \quad . \quad . \quad . \quad (2)$$
$$x\Sigma(ac) + y\Sigma(bc) + z\Sigma(c^2) = \Sigma(cm) \quad . \quad . \quad . \quad (3)$$

We have thus at length obtained three final equations—(1), (2), and (3)—which will enable us to determine the most probable values of x, y, z. The following example, taken from Merriman, will serve as a numerical illustration of this method. Suppose we have the four following equations for determining three unknown quantities—

$$\begin{aligned} x - y + 2z &= 3 \\ 3x + 2y - 5z &= 5 \\ 4x + y + 4z &= 21 \\ -x + 3y + 3z &= 14. \end{aligned}$$

Then multiplying by the coefficients of x we obtain the following series :—

$$\begin{aligned} x - y + 2z &= 3 \\ 9x + 6y - 15z &= 15 \\ 16x + 4y + 16z &= 84 \\ x - 3y - 3z &= -14 \end{aligned}$$

Hence by addition $\quad 27x + 6y = 88 \quad . \quad . \quad . \quad (1)$

In like manner we obtain $\quad 6x + 15y + z = 70 \quad . \quad . \quad . \quad (2)$

And finally $\quad y + 54z = 107 \quad . \quad . \quad . \quad (3)$

From which we find $x = 2{\cdot}4702$, $y = 3{\cdot}5509$, $z = 1{\cdot}9157$. These

values will not exactly satisfy the equations, but give us the following residuals :—

$$\delta_1 = -\ 0\cdot2493$$
$$\delta_2 = -\ 0\cdot0661$$
$$\delta_3 = +\ 0\cdot0945$$
$$\delta_4 = -\ 0\cdot0704$$

the sum of whose squares is 0·0803. This quantity will be found to be less than the sum of the squares of the residuals derived from any other values of x, y, z.

For a good illustration of the use of this method in discussing physical observations the student is referred to a "Report of a Magnetic Survey of Scotland" by General Sabine, which will be found in the *Transactions of the British Association* for 1836.

15. *The Graphical Method.—Treatment of Observations connecting together Simultaneous Values of two Physical Quantities.*—Let us take the example already given, and suppose that we have obtained values of the maximum pressure of aqueous vapour for certain definite temperatures between 0° and 40° C., and that our object is to construct a table by means of these values. Probably our best method will be to obtain some curve paper, or paper ruled in small squares by means of lines at regular intervals from each other—say, for example, one-tenth of an inch, one-twentieth of an inch, or one millimètre.

Let us suppose that we represent temperatures between 0° and 40° by our line of abscissæ, and that ordinates are raised to denote the corresponding maximum pressures which we have obtained in our experiments.

Now, taking the following table as embodying the actual result of our experiments :—

Recorded Temperatures.	Recorded Maximum Pressures in inches.
0	0·141
8·5	0·386
17·2	0·605
22·3	0·758
28·4	1·092
35·4	1·886

let us "plot a curve" after the manner of Lesson XXIII.

We shall soon see that the line that we have attempted to

draw accurately through the indicated points contains irregularities which do not represent natural facts, and which we can only imagine to be due to errors of experiment. What we have now to do is to pass a curve line evenly between our observations, this line being likewise such that the eye, glancing tangentially along it, is satisfied with its smoothness and symmetrical course.

Such a line will represent the best method of exhibiting the results of our observations, and will very frequently afford us the means of detecting experimental errors. Suppose, for instance, that, as in Regnault's case, our temperature range were to extend from $-32°$ C. to $+230°$ C., our object being to determine a table of maximum pressures of aqueous vapour, and that, during the course of our experiments, two different instrumental methods were employed—one for temperatures below 100° C., and the other for those above that point. If there were no twist or want of symmetry in the progress of our curve as it passed this turning point we should be entitled to argue that the change from the one experimental method to the other had made no sensible difference in the value of our results.

If, however, there were a peculiar twist which we could not imagine to represent a natural law, we might conclude that there was some error in our experiments, and that our two methods did not agree together.

So delicate is this graphical method of detecting error that we might by its means very readily detect errors in tables of logarithms or trigonometrical functions. But while we bring a smooth curve evenly between our experiments, we must be very careful that we do not employ this smoothing process to such an extent as to cloak or exclude the indications of some physical law.

It is imagined that Regnault, a very accurate observer, committed a mistake of this kind in his discussion of his observations of the maximum pressures of aqueous vapour for various temperatures. In discussing these he used the graphical method, and obtained a curve from which it appeared to him that the passage of water from the liquid to the solid state was without influence upon the vapour densities ; in other words, his curve appeared

to show no peculiar point at 0° C. Professor James Thomson has, however, shown that in all probability the ice-steam curve, or that below 0° C., is slightly different from the water-steam curve, or that above 0° C., and the observations of Ramsay and Young have confirmed this result. We must, therefore, be very careful not to cloak a real law by our graphical method.[1]

16. *Interpolation.*—It is hardly necessary to remark that having obtained a curve such as that we have just supposed to have been plotted, embodying the result of our observations, it is very easy to obtain from this curve the maximum pressure for any particular temperature. This process is called "interpolation." The process is of great use in the formation of tables. Generally we can interpolate best by the graphical method just mentioned, but sometimes it may not be convenient or possible to apply this, and we must then resort to some *interpolation formula.* Let us suppose, for example, that we have a table of the squares of the numbers from 10 to 16, and that we take the differences of these as under—

Number.	Square of number (Y).	First set of differences (Δ').	Second set of differences (Δ'').	Third set of differences (Δ''').
10	100			
		21		
11	121		2	
		23		0
12	144		2	
		25		0
13	169		2	
		27		0
14	196		2	
		29		0
15	225		2	
		31		
16	256			

[1] It may be asked, How can we obtain from the curve a numerical expression of the law connecting the given quantities? Where the line is sufficiently straight to be taken as a straight line, the required expression is in the form of an equation to a straight line, namely,

Suppose next that we wish to find the square of 17. Let us take 14 as our starting-point, or zero. Now the difference between 17 and 14 is 3. Let us therefore make $x = 3$, and apply the following formula[1]—

$$Y_x = Y_0 + x\Delta'_0 + \frac{x(x-1)}{1.2}\Delta''_0 + \frac{x(x-1)(x-2)}{1.2.3}\Delta'''_0 + \text{etc.}$$

Here $\quad Y_0 = 14^2 = 196$; also $\Delta'_0 = 29$, $\Delta''_0 = 2$, $\Delta'''_0 = 0$.

Hence $\quad Y_3 = 17^2 = 196 + (3 \times 29) + (3 \times 2) + 0 = 289.$

In like manner, if we wished to find $(17{\cdot}4)^2$ we should make $x = 3{\cdot}4$, and hence

$$(17{\cdot}4)^2 = 196 + (3{\cdot}4 \times 29) + (4{\cdot}08 \times 2) = 302{\cdot}76.$$

Had we taken 13 as our starting-point, we should have had in like manner

$$Y_4 = 17^2 = 169 + (4 \times 27) + (6 \times 2) + 0 = 289 \text{ as before.}$$

In like manner we could find $(12{\cdot}2)^2$, $(15{\cdot}8)^2$, etc.

In order to apply this method to a physical formula, suppose that we have values of the maximum pressure of aqueous vapour at various temperatures, and that we obtain differences of these until they vanish, or become so small as to be negligible, as follows—

Temperature cent.	Pressure in millimètres.	Δ′.	Δ″.	Δ‴.
42	61·055			
		3·291		
43	64·346		0·153	
		3·444		0·004
44	67·790		0·157	
		3·601		
45	71·391			

$y = mx + c$, where m is the tangent of the angle the line makes with the axis of x, and c is the ordinate at the origin. If the line cannot be regarded as straight, and cannot be connected with any known equation, the determination will present difficulties that cannot be entered into here.

[1] A proof of this formula will be found in Galbraith and Haughton's *Manual of Algebra*.

Using 42° as our zero or starting-point, it is required to find the maximum pressure of aqueous vapour for 46°.

Here $Y_4 = 61\cdot055 + (4\times3\cdot291) + (6\times0\cdot153) + (4\times\cdot004) = 75\cdot153$.

The true number as given by the table is 75·158.

If the maximum pressure at 42°·5 be required, we should have

$$Y_{\cdot5} = 61\cdot055 + (\cdot5\times3\cdot291) + \frac{(\cdot5)(-\cdot5)}{2}(\cdot153) + \frac{(\cdot5)(-\cdot5)(-1\cdot5)}{6}(\cdot004)$$

$$= 61\cdot055 + 1\cdot6455 - \cdot019125 + \cdot00025 = 62\cdot682.$$

Frequently the terms arising from the second and third order of differences may be neglected. We see examples of the application of interpolation so simplified in the case of logarithmic tables, and in the laboratory in using the spectroscope, Wheatstone's bridge, and other apparatus.

B.

CENTIMÈTRE—GRAMME—SECOND (C. G. S.) SYSTEM OF UNITS.

The things which come most prominently before us in physical science are extension,—including length, surface, and volume,—time, mass, velocity, force, work. For each of these we must select a unit or standard, in terms of which all other values of the thing in question must be numerically expressed.

Thus we speak of a thing being 6 *feet* long, or of a time equal to 11 *seconds*, and so on.

The units of length, time, and mass are *fundamental units*, while those of surface extent, volume, velocity, force, and work are *derived units*. Presuming that we are free to choose any unit of *length* we think proper, yet, if we have once chosen the *centimètre* as our unit of length, it would be highly inconvenient to choose any other unit of *area* than the *square centimètre*, or any other unit of *volume* than the *cubic centimètre*. Again,

we are perfectly free to choose our unit of *time*, but having once chosen the *second*, it would be highly inconvenient to choose any other unit of *velocity* than that which denotes *one centimètre moved over in one second.*

In like manner we are perfectly free to select our unit of *mass*, but having once chosen the *gramme*, it would be highly inconvenient to select any other unit of *force* than *that which when it has acted for unit of time* (*one second*), *or unit of mass* (*one gramme*), *shall have produced unit of velocity* (*that of one centimètre in one second*).

Finally, the appropriate unit of *work* will, in like manner, be defined as *that spent in overcoming unit of force through unit of length.*

In the C. G. S. system the unit of force is termed the *dyne*, and the unit of work the *erg*, so that we have finally the following consistent definitions :—

(1) The unit of *length* is one centimètre.
(2) The unit of *area* is one square centimètre.
(3) The unit of *volume* is one cubic centimètre.
(4) The unit of *time* is one second.
(5) The unit of *mass* is that of one cubic centimètre of distilled water at its maximum density, which is called the *gramme.*
(6) The unit of *velocity* is that of one centimètre in one second.
(7) The unit of *force* or *dyne* is the force which will produce a velocity of one centimètre per second in a free mass of one gramme by acting on it for one second.
(8) The unit of *work* or *erg* is that spent in overcoming the force of one dyne through the length of one centimètre.

A few simple examples will show the student how to express physical values in terms of the (C. G. S.) system.

Example I. What is the value in the C. G. S. system of the force represented by the weight of 5000 grains at Greenwich and by that of 485 grammes at Paris at the level of the sea ?

Answer. It will be seen by reference to Table W that the

acceleration produced by gravity at Greenwich is 981·17 centimètres, while at Paris it is 980·94 centimètres. Hence

weight of 5000 grains at Greenwich

$$= 5000 \times \cdot064799 \times 981\cdot17 = 317894\cdot2 \text{ dynes};$$

weight of 485 grammes at Paris

$$= 485 \times 980\cdot94 = 475755\cdot9 \text{ dynes}.$$

Example II. What is the value in the C. G. S. system of the velocity of a mile per hour?

Answer. It will be

$$\frac{5280 \times 30\cdot479449}{3600} = 44\cdot7032.$$

Example III. What is the value in the C. G. S. system of one foot-pound at Manchester, and of one kilogramme-mètre at Paris, the former meaning the work spent in raising one pound one foot high against gravity at Manchester, and the latter the work spent in raising one kilogramme one mètre high against gravity at Paris?

Answer. The former will be

$$30\cdot479449 \times 453\cdot593 \times 981\cdot34 = 13567285\cdot3 \text{ ergs},$$

where the first multiplier is on account of the distance, the second on account of the mass used, and the third on account of the local value of gravity.

Again, the latter will be in like manner

$$1000 \times 100 \times 980\cdot94 = 98094000 \text{ ergs}.$$

Example IV. What is the value in *ergs* of the visible energy of a mass of 123 grammes which is moving with the velocity of 65 centimètres per second?

Answer. Let us make use of the principle by virtue of which a body of unit mass projected upwards against gravity ($g = 32$) with the velocity of 32 feet per second will rise 16 feet, while if projected upwards with the velocity of 64 feet per second it will rise four times as high, or 64 feet. Hence we see that a body of the mass of one *gramme*, having the velocity of one centimètre per second, will overcome the force of one *dyne* through the distance of half a centimètre, and be thus capable of doing half a unit of work, and also that the capability of doing work will vary with the square of the velocity. From

this it follows that the amount of energy possessed by the body in question will be represented by

$$\frac{123 \times 65^2}{2} = 259837 \cdot 5.$$

These examples are sufficient to illustrate the system. For further information the student is referred to Professor Everett's *Units and Physical Constants.*

"*Weight*" *and* "*Mass.*"

By the *weight of a body* is *strictly* meant the **force** which tends to cause the body to move downwards. The amount of matter in a body is called its **mass**, a quantity that is measured in terms of a piece of metal called a **Standard Mass**, taken as a unit. Masses may be compared by studying the action on them of *any force whatever.* No name has been given to this process of comparison, but since, in everyday life, the force which tends to move the body downwards is made use of to effect a comparison, the process is usually called "weighing." This process depends upon the fact that at a constant locality *weight is directly proportional to mass*, so that if a mass is doubled its weight will be doubled, and when one is increased the other will be increased in a like proportion.

A force when acting on a body is measured by the *product of the rate of change of velocity* (acceleration) *produced, and the mass of the body moved.* Thus, if a body of weight W and mass M, when free to fall, move downwards with an acceleration g, we shall have

$$W = Mg.$$

If C. G. S. units be selected, the weight will be in *dynes.* The weight in dynes of a body, we thus see, depends upon the value of g. But the value of g differs with the locality, and this must always be stated in any definition of weight.

When the student is quite clear about these points, he may, without fear of confusion, use the ordinary phraseology, and speak of "weighing" in the sense of comparing masses, and of "weights" when "masses" are meant.

C.

PRACTICAL NOTES.

1. *Steel Scales.*—Scales on steel, ruled to any pattern, may be obtained from several makers, as Whittam of Salford and Chesterman of Sheffield. If nickel-plated they are protected from rust, but this is not recommended if the scales are much in use, the nickel having a tendency to chip off. Should they become rusty they may rapidly be cleaned by fastening them down on a table and rubbing them with emery cloth wrapped round a block of wood. Excellent paper scales may be printed at a small cost by lithography from steel scales, but in this case the latter should be without numerals, or be engraved in a reversed manner—that is, from right to left, as seen in a mirror.

2. *Cross-wires or Cross-threads.*—We have used these terms in a general sense for the reference lines placed in the focus of the eye-piece of a telescope. As a rule each cross-wire consists of a single thread from the web of the spider. To replace these threads when broken requires much patience, but the operation is much less difficult than might at first be thought. Everything depends upon obtaining a supply of the proper kind of web, that of the common garden-spider being the best. This little spider, which may easily be caught in the summer from a hedge, gives a strong web, differing in fineness with the size of the insect. To obtain the thread fit up a simple machine consisting of a piece of wire bent with two prongs, say $1\frac{1}{2}$ inch apart (a hair-pin will serve the purpose very well); mount this upon a bobbin in such a way that the wire may be easily revolved. If the spider be placed upon the wire and the latter tapped, the insect will probably attempt to escape by spinning one of its threads. As it spins the thread the wire may be revolved and the thread wound across the prongs. With a little manipulation a considerable quantity of the web may be collected and preserved for future use.

To insert cross-wires in a telescope the diaphragm must be removed and guide lines be ruled on it to help in fixing the threads at right angles. One of the threads may then be laid across and very slightly stretched, its exact position being adjusted by a hair. A very small quantity of shellac varnish of the proper consistency will then fix it securely. The second thread must be fixed in the same way; but this operation will present greater difficulties, since it will now be necessary to have the two threads at right angles to each other.

As a substitute for the spider threads cocoon silk may be used, but it is usually too coarse for the magnification of the eye-piece. Two lines ruled upon glass with a diamond is sometimes used, and platinum wire of extreme fineness has been proposed.

3. *Cathetometer Adjustments.*—Some additional details relating to the adjustments of the cathetometer will here be given. These adjustments being very similar to those that have to be applied to the surveying theodolite, manuals relating to surveying instruments should be consulted for still further details. (1) Adjustment I., page 31, namely, to bring the centre of the cross-wires to coincide with the optical axis of the telescope, is technically known as *collimation*. It may best be performed in the following systematic manner. Fix vertically a finely-divided scale some distance from the cathetometer, focus the telescope upon it, and adjust the horizontal cross-wire *bd* (Fig. 13*a*) upon one of the divisions, then rotate the telescope through 180° about its own axis. Read the position that *bd* has now upon the scale. Adjust *a* or *c* until the horizontal cross-wire is half-way between the two observed positions. Again rotate and take out half the error. Continue this until the horizontal wire does not change its position on rotation through 180°. Now make *ac* horizontal and adjust as before. The centre should now be in perfect adjustment. (2) If the axis of the cathetometer be *first made vertical* by Adjustment III., then Adjustment II. may be made quite independently of the level. To do this, focus the telescope upon a distant vertical scale. Now remove the telescope from the Y's and turn it end for end. The telescope will now be pointing away from the scale, and in order

to bring the latter again into view the cathetometer must be revolved about its own axis. On again observing the mark, if the centre of the cross-wires falls exactly upon the same place as before we may be sure that the axis of the telescope is horizontal. Should this not be the case take out half the error by the telescope levelling-screw and repeat the operations until the required condition is secured. On now observing the level it may be found to be out of adjustment. It will then only be necessary to make use of the screw belonging to the level to bring the bubble between the index-marks in order to complete the preliminary adjustment. It may be necessary to repeat several times this series of adjustments.

4. *Silk Fibre.*—For the suspension of bodies during density determinations fine silk thread made up of about eight cocoon-fibres is suitable. It is known as "raw silk," and, when good, a single thread will support 50 grammes, but should the thread be unduly stretched, the tenacity will be much diminished.

5. *Platinum Wire.*—It is easy to obtain platinum wire for density determinations so fine that a mètre of the wire capable of carrying 50 grammes will weigh only ·06 gramme. If 100 mm. of the wire be used in a density determination it will weigh ·006 gramme, and lose in water ·0003 gramme, and so only affect the fourth place of decimals. Such wire requires care in handling, and must be kept as straight as possible to avoid *kinks*, which cause the wire to break.

6. *Drying Specific Gravity Bottles.*—Where time is of importance, or a drying oven is not available, small bottles may be rapidly dried in the following way :—Take a piece of hard glass tubing about 200 mm. (8 inches) long, and about 10 mm. (·4 inch), and draw out one end so that it may be sufficiently small to insert in the bottle. Bend the tub at right angles at about 50 mm. (2 inches) from this end. Fix the tube in a clamp with the bent end vertical, and allow the horizontal portion to be heated by a large Bunsen flame. Now, by means of india-rubber tubing attached to the large end of the glass tube and to

the bellows of a glass blow-pipe, cause a gentle current of hot air to pass into the bottle, which must be placed resting mouth downwards on the upright portion of the glass tube.

7. *Mercury.*—Mercury in a tolerably pure state is imported into this country in iron bottles containing 70 lbs. of the liquid metal. In this state it is sufficiently free from other metals for most laboratory purposes. It will, however, require to be freed from dust by the method generally adopted.—A piece of clean writing-paper is folded twice and then opened out in the form of a cone ; it is then fitted into a glass funnel. A few holes are made with a fine needle near the apex of the paper cone, so that when mercury is poured in the liquid escapes through these in fine streams. Mercury filtered in this way is quite free from dust. Much care should be taken with mercury lest it should become contaminated with lead, zinc, and other metals with which it readily amalgamates. The presence of these metals may be recognised at once by the mercury leaving a *tail* of dirt when it is made to run down a white porcelain dish. The purification of mercury may be conducted as follows :—A solution of mercurial nitrate is prepared by dissolving mercury in nitric acid, and is added to the impure mercury, which should be contained in a shallow porcelain dish. The action of the liquid is to give at once a perfectly pure appearance to the mercury ; but it will be necessary to leave the materials in contact for several weeks, stirring from day to day, before the metallic impurities will entirely leave the metal and enter into solution. When the process is judged to be complete the mercury should be poured into a separating funnel,—*i.e.*, a funnel provided with a stopcock,—so that the heavy metal may be run off and separated from the water above. To obtain mercury quite pure redistillation is necessary, the process being conducted at a low temperature by partially exhausting the air of the retort and receiver.

8. *Curves and Curve Paper.*—Paper ruled in squares may be obtained from several stationers in London, such as Williams and Norgate, Waterlow and Sons, and Letts. The most useful kinds are those ruled in centimètres and millimètres, inches and

twentieths of an inch. The paper should have a good surface, and the ruling be so clear that the different lines may not be confused. Before making use of it for a curve a suitable scale must be chosen. This will depend upon the size of paper used, the extent of the observations, the number of places of figures to which the results have been carried, and the relation which it is desired to make most apparent. Usually an accuracy up to three significant figures can only be obtained when the scale is very large and the greatest pains are taken. The observed points of the curve may be marked with small dots made with a hard sharp pencil, and these may be included in small circles, or marked by small crosses, or stars (see page 82). In drawing the curve the successive points must first be joined by thin straight lines, and a smooth curve then drawn through them in a free-hand manner.

9. *Glass Working.*— This valuable accomplishment to the worker in the physical laboratory can only be attained by great practice and personal instruction. We shall therefore confine our observations to a few useful hints. (1) *Use of file.*—A small triangular file is used for separating lengths of glass tubing. With narrow tubes a single scratch is made, and the tube is then bent inwards from the scratch, at the same time pulling the glass on either side of the scratch in opposite directions, when the tube readily divides. The more the pull and the less the bend the cleaner the division. Larger tubes must be filed partially round. (2) *Grinding and shaping glass.*—The edges of glass plates may be ground smooth by using a plate of glass with fine emery and water or turpentine. Rough grinding is quickly done on a common grindstone. Corners of glass plates may be removed by a flat pair of nippers used so as to break away fragments of glass gradually. A key may be usefully employed and with less danger of long fractures in the same way. Insert the glass between the wards of the key, which must then be worked backwards and forwards so that the glass is frittered away. This is a good method for shaping a thick tube. (3) *Use of diamond and substitute.*—For the clean and easy cutting of sheets of glass a diamond is indispensable, but in its absence the cheap substitute in which a small wheel of extremely hard

steel replaces the diamond may be employed. (4) *Use of heat.*—The bottom of beakers or flasks may be cut out very conveniently by a small gas flame obtained by the use of a glass tube with small opening. Start a crack by heating and sudden cooling of the glass, then by applying the flame a little in advance of the crack the latter may be led in any required direction. In using the method it is always found that when the crack approaches the starting-point, as when leading the crack round a beaker, it cannot be made to continue beyond perhaps $\frac{1}{4}$ inch of the part already cracked. It must therefore be broken off at this point, and the little projection removed by the file. Tubes of a diameter as great as 2 or 3 inches may be cleanly broken across by the following process: Tie two separate pieces of string firmly round the glass on each side of the place where it is desired to divide the glass, leaving a space between the folds of twine. Now pass once round the space a strong tightly-twisted piece of cord, and let two persons take hold each of one end of the cord and rapidly draw the cord first one way and then the other, the tube meanwhile being held firmly. The friction of the cord will cause the glass to become quite hot, so that if water be thrown upon it the tube will break clean across at the desired place. (5) *Boring glass.*—To drill a hole in glass use a small hand-drill—watchmakers' or Archimedean drill—the glass being kept wet with turpentine. The pressure must be very gentle when the glass is nearly pierced. (6) *Use of blowpipe.*—The kind of blowpipe needed is a Herepath blowpipe for use with gas and a foot bellows. Before submitting glass to the full flame it must be gradually and uniformly heated. (*a*) To round the end of a tube, simply hold it in the flame, at the same time rotating. (*b*) To draw out a tube, hold horizontally in both hands, slowly approach the flame, at the same time rotating the tube, this operation being continued while the tube is in the flame. When the glass is felt to be sufficiently soft remove it from the flame, and draw it out more or less quickly according to tapering required. (*c*) To close the end of a tube, heat one end and press it against the already heated end of a second piece of tube, so that the two tubes may adhere. Heat the first tube at a little distance from the joint, then draw out so

as to close its end. Heat the tapering end and remove superfluous glass by touching it with a piece of glass. Continue the heating, and quickly apply the open end to the mouth and blow out the end round. (*d*) If a bulb is required at the end of a tube the process is as in *c*, the main thing being to collect sufficient glass to provide a bulb of the size required, with walls of sufficient thickness. If too much glass is used, a knob will be produced at the end of the bulb. The bulb when blown should be gradually cooled, so as to anneal it. (*e*) Quill tubing may best be bent in an ordinary bat's-wing flame, the blowpipe or Bunsen being too hot for the purpose. All bends should be gradual. A sudden bend produces a wrinkling ; wrinkled tubes generally crack. The bending of tubes of large diameter by the blowpipe requires very cautious heating, not only at the place where the bending is required, but for some length on each side.

10. *Apparatus for Pendulum Experiments.*—A wooden stand fixed up in front of an ordinary clock with seconds pendulum has been found to be very convenient for pendulum experiments. The stand is $8\frac{1}{2}$ feet high, $1\frac{1}{2}$ feet broad, and has two shelves 6 inches wide, which may be moved and clamped in any position. The upper shelf has the pendulum bearings, which are placed one on each side of a square hole cut in the centre of the upper shelf. These bearings are made in the following way :—A bar of steel 5 inches long, $\frac{1}{2}$ inch breadth, $\frac{3}{4}$ inch thickness, is cut into two. The two pieces are laid side by side, and small holes drilled to admit screws. They are then strongly screwed together with their $\frac{1}{2}$-inch faces together, and a hole $\frac{3}{4}$ inch in diameter is drilled so that when the pieces are separated a semicircular groove will have been formed at the middle of each $\frac{1}{2}$-inch face in each piece of steel. The knife-edge for the simple pendulum may be made out of an old three-cornered file, which must be softened, one of the edges filed flat, and a small hole drilled through vertically to the base of the triangle. The flattened edge is then filed sharp and the steel hardened and polished. The wire to support the pendulum is passed through the hole in the knife-edge and secured. The bob of the pendulum is a ball of brass about 2 inches diameter, and

there is a small screw to fit a hole in the ball. The small screw is pierced with a fine hole so that the lower end of the suspending wire may be passed through it and fastened. The screw when in position lies with its head flush with the top of the ball. In this way the correction for moment of inertia is simplified. The error due to the knife-edge may be estimated if desired; but the error may, as a rule, be neglected, or, better still, eliminated altogether by providing the knife-edge with Borda's arrangement. For obtaining the flashes of light a hole should be bored in the side of the clock-case to admit the light from a paraffin or Argand lamp. A small mirror is fixed to the clock pendulum in such a way that the inclination of its plane to the light may be adjusted. A small screen of blackened wood placed outside the clock-case, provided with an adjustable slit, is necessary for altering the size of the flash of light.

11. *Division of a Length into Equal Parts.*—Lesson VII. gave the method of using the dividing engine for this purpose. The operation may be effected by simpler methods. (*a*) When the number of parts is even it is only necessary to continually bisect the length. When the number of parts is odd the method becomes one of trial and error, using submultiples of the length to check the work. Spring bows with fine needle-points must be used for the purpose. (*b*) On a sheet of hard glass etch a scale such as has been described for measuring with the electro-chronograph (Lesson LII.), in which a number of lines radiating from a point intersect a line divided into equal parts. Suppose that we wish to divide a line of 10 mm. into eleven equal parts. We move a millimètre scale with bevelled edge parallel to the line AB (see Fig. 64, in which AB is supposed equally divided) until a place is found where a line of 10 mm. is divided into eleven equal spaces. Let the scale be fixed at this point; the divisions may then be transferred from the glass to paper by compasses, or if an etched scale is desired they may be transferred to a waxed slip, as in Lesson VIII. (*c*) A third method is to use **Proportional Compasses**, such as are provided with mathematical instruments. (*d*) A **Sector** or jointed rule, with lines radiating from the joint, may also be employed. A pair of the

lines marked on the rule with the letter L are called the *line* of *lines.* Suppose that we require to divide a line into seven equal parts. Take the length by compasses, and open the rule until the distance from 7 to 7 of the line of lines is equal to the distance between the compass-points. Keeping the limbs of the rule at the same angle, measure off the distance 4 to 4, which will be equal to $\frac{4}{7}$ of the line; set off this distance from *both ends* of the line. The distances between the two ends will overlap to the extent of $\frac{1}{7}$ of the line; it remains, therefore, only to divide the line into spaces of this length. (*e*) A special kind of sector with a sliding scale has been designed by Miss Marks, and made by W. F. Stanley, called the **Line Divider,** which does not require the use of compasses, but may be directly applied to the line that is under division.

12. *Dense Liquids.*—To prepare the solution of double iodide of potassium and mercury dissolve 271 parts by weight of corrosive sublimate (mercuric chloride) in water, and add to it a solution containing 332 parts by weight of potassium iodide. Filter the liquid in order to obtain the precipitated iodide of mercury, which must be dissolved in a saturated solution of potassium iodide containing 332 parts by weight. The solution obtained is then evaporated on a water-bath to obtain a liquid of the maximum density, which may be diluted as required. The double iodide of barium and mercury may be prepared in the following manner. Place 100 parts by weight of barium iodide and 130 parts of mercuric iodide in a dry flask, shake well, and then add 20 parts of water. Heat the mixture in an oil-bath to 150° C., and shake well until all is dissolved. After concentration on a water-bath, preserve the solution in a well-stoppered bottle. For purposes of dilution a weaker solution will be required.

END OF VOL. I.

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